

Fibration

Thursday, April 12, 2018 2:02 PM

Two examples of model categories

- ① Topological spaces (pointed or not)
- ② Chain complexes of R -modules
for a ring R

Def A model category \mathcal{M} is a category with \rightarrow classes of morphisms

- a) WEAK EQUIVALENCES (WEQ)
- b) FIBRATIONS
- c) COFIBRATIONS

satisfying \mathcal{M} axioms.

In ① a weq is a map inducing an isomorphism of π_1 groups for all base points. Examples of cofibrations include $A \hookrightarrow B$ where A is a closed subspace of B .

~~Simon~~

~~category of sets~~

③ $\mathcal{M}_0 = \text{Set} = \text{category of sets}$

a) WEQs are isomorphisms

all maps are fibrations and cofibrations

b) Fibrations are iso, and all maps are WEQs and cofibrations

c) Dual

MCI \mathcal{M} is closed under limits and colimits.

Easy consequences

a) \mathcal{M} has an INITIAL OBJECT ϕ i.e. an object with a unique morphism to any other object.

b) \mathcal{M} has a TERMINAL OBJECT $*$, i.e. an object with a unique morphism from any other object.

A model category is POINTED if the initial and terminal objects are the same.

c) \mathcal{M} has products + coproducts
 (Cartesian products + disjoint unions
 in Set)

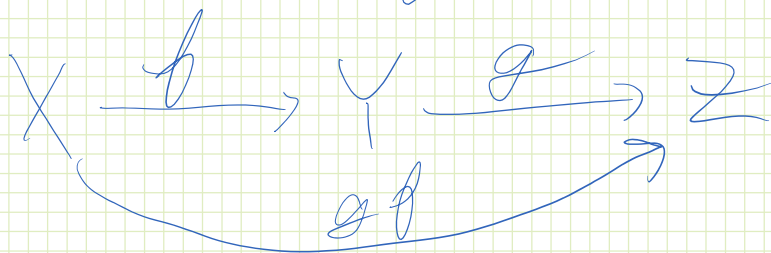
d) Suppose we have $W \xrightarrow{i} X \xrightarrow{f} Y \xrightarrow{j} Z$ in \mathcal{M} .

~~Suppose we have~~

$\exists! W' \xrightarrow{\alpha} W \xrightarrow{i} X$ with $i\alpha = i'$.
 W is the EQUALIZER of f and g .

There is also a COEQUALIZER
 $Y \xrightarrow{j} Z$ with $jf = jg$ etc.

MC2 Two-out-of-three. Given



If any 2 of f, g and gf are
 WEQs, so is the third.

MC3. Retract AXIOM

A is a RETRACT of B if there are maps

$$\begin{array}{c}
 A \xrightarrow{i} B \xrightarrow{m} A \\
 \underbrace{\hspace{10em}}_A \uparrow
 \end{array}$$

f is a retract of g if $\exists i, i', m$ and m' s.t. $mi = 1_A$ and $m'i' = 1_{A'}$

$$\begin{array}{ccccc}
 A & \xrightarrow{i} & B & \xrightarrow{m} & A \\
 \downarrow f & & \downarrow g & & \downarrow f \\
 A' & \xrightarrow{i'} & B' & \xrightarrow{m'} & A'
 \end{array}$$

and $fm = i'g$ and $gi' = i'f$.

A retract of a WEQ is a WEQ

Fibration
 Cofibration cofibration

~~THE LIFTING AXIOM~~ AXIOM. Suppose

$$\begin{array}{ccc}
 & & \downarrow p \\
 i \downarrow & \xrightarrow{h} & \downarrow p \\
 B & \xrightarrow{\quad} & Y
 \end{array}$$

(1) i is a cofibration, and p is both a fibration and a WEQ
 or (2) i is a cofibration and a WEQ, and

p is a fibration.

A map which is a fibration + and a WEQ is a ^{trivial} (ACYCLIC) fibration

|| COFIBRATION

MC5 FACTORIZATION (the hard part)

