Fibration

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Two examples of model categories

① Topological spaces (pointed or not)

② Chain complexes of R-modules (nonnegatively graded) for a ring R

Def A model category $\mathcal{M}$ is a category with 3 classes of morphisms

a) WEAK EQUIVALENCES (WEQ)

b) FIBRATIONS

c) COFIBRATIONS

satisfying 5 axioms . . . .

In ① a weq is a map inducing an isomorphism of htpy groups for all base points. Examples of cofibrations include $A \hookrightarrow B$ where A is a closed subspace of B. .

[illegible]

③ $\mathcal{M}$ = Set = category of sets

a) WEQS are isomorphisms
all maps are fibrations and cofibrations
b) Fibrations are iso, and all maps
are WEQs and cofibrations
c) Dual

MC1 $\mathcal{M}$ is closed under limits
and colimits.

Easy consequences
a) $\mathcal{M}$ has an INITIAL OBJECT $\emptyset$
i.e. an object with a unique
morphism to any other object.
b) $\mathcal{M}$ has a TERMINAL OBJECT $*$,
i.e. an object with a unique
morphism from any other object.

A model category is POINTED if
the initial and terminal objects are the
same.

c) $\mathcal{M}$ has products + coproducts (Cartesian products + disjoint unions in Set)

d) Suppose we have $W \xrightarrow{i} X \overset{f}{\underset{g}{\rightrightarrows}} Y \xrightarrow{j} Z$ in $\mathcal{M}$.

$\exists !\, W' \xrightarrow{\alpha} W \xrightarrow{i} X$ with $i\alpha = i'$.

W is the EQUALIZER of f and g.

There is also a COEQUALIZER $Y \xrightarrow{j} Z$ with $jf = jg$ etc.

MC2 Two-out-of-three. Given

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

If any 2 of f, g and gf are WEQs, so is the third.

MC3. Retract AXIOM

A is a RETRACT of B if there are maps

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[rr, bend right, "A"'] & B \arrow[r, "r"] & A \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[r, "r"] \arrow[d, "g"'] & A \arrow[d, "f"] \\ A' \arrow[r, "i'"] & B' \arrow[r, "r'"] & A' \end{tikzcd}$$

f is a retract of g if $\exists$ i, i', r and r' s.t. $ri = 1_A$

$r'i' = 1_{A'}$

and $fr = r'g$ and $gi = i'f$.

A retract of a WEQ is a WEQ

Fibration

Cofibration cofibration

LIFTING AXIOM. Suppose

$$\begin{tikzcd} {} \arrow[d, "i"'] & & {} \arrow[d, "p"] \\ B \arrow[rr, "\beta"] \arrow[urr, dashed, "h"] & & Y \end{tikzcd}$$

① i is a cofibration, and p is both a fibration and a WEQ

or ② i is a cofibration and a WEQ, and p is a fibration:

a map which is a fibration + and a WEQ is a trivial (ACYCLIC) fibration

" COFIBRATION - - - - -

MC5 FACTORIZATION (the hard part)

$$
\begin{tikzcd}
 & \hat{X} \arrow[dr, "\text{fibration}"] & \\
X \arrow[ur, "\text{trivial cofibration}"] \arrow[rr, "f"] \arrow[dr, "\text{cofibration}"'] & & Y \\
 & \hat{Y} \arrow[ur, "\text{trivial fibration}"'] &
\end{tikzcd}
$$