Thursday, April 19, 2018 1:56 PM

Recall: for a model category $\mathcal{M}$ (e.g. Top or $\mathcal{J}$) and a small category $\mathcal{J}$ we can define the category $\mathcal{M}^{\mathcal{J}}$ of functors $\mathcal{J} \to \mathcal{M}$. For such a functor X, we denote its value on an object $j \in \mathcal{J}$ by X_j and for a morphism $j \xrightarrow{f} k$ in $\mathcal{J}$, X_f denotes the map $X_j \to X_k$. Since X is a functor $\mathcal{J} \to \mathcal{M}$, we get structure maps

$$\boxed{\mathcal{J}(j,k)} \times X_j \xrightarrow{\varepsilon^X_{j,k}} X_k$$

$\|$

set of morphisms $j \to k$ in $\mathcal{J}$.

Example Let $\mathcal{J}$ is the category

$$0 \to 1 \to 2 \to 3 \to 4 \to \cdots$$

A functor $\mathcal{J} \xrightarrow{X} \mathcal{M}$ is a diagram

$$X_0 \to X_1 \to X_2 \to X_3 \to \cdots \quad \text{in } \mathcal{M}.$$

Recall a spectrum X is a collection of pointed spaces $\{X_n : n \geq 0\}$ with structure maps $S^1 \wedge X_n \xrightarrow{\varepsilon_n^X} X_{n+1}$

and hence $S^k \wedge X_n \xrightarrow{\varepsilon^X_{n,n+k}} X_{n+k}$

We would like to describe a spectrum as a $\mathcal{T}$-valued functor on some "category" $\mathcal{J}$ whose objects are nat #s n with "morphism sets"

$\mathcal{J}(n, n+k) = S^k$, a topological space

Digression on ENRICHED CATEGORY THEORY.

Recall a category $\mathcal{C}$ consists of

- a collection of objects
- for each pair (a_1, a_2) of objects we have a set $\mathcal{C}(a_1, a_2)$ of morphisms $a_1 \longrightarrow a_2$
- to compose morphisms $a_1 \longrightarrow a_2 \longrightarrow a_3$

$$\mathcal{C}(a_2, a_3) \times \mathcal{C}(a_1, a_2) \longrightarrow \mathcal{C}(a_1, a_3)$$

IDEA Suppose $C(a_1, a_2)$ is an object in a category V rather than a set.
V must be MONOIDAL, i.e. there must be a funcitor $V \times V \xrightarrow{\square} V$ and a unit object 1, so we have $(V, \square, 1)$

Examples $(Set, \times, *)$, $(Set, \amalg, \emptyset)$ $(Top, \times, *)$, $(\mathcal{J}, \wedge, S^0)$, $(Ab, \oplus, 0)$, $(Ab, \otimes, \mathbb{Z})$

A monoidal catory $(V, \square, 1)$ is SYMMETRIC if $a \square b \cong b \square a$ (natural iso)

Def For a symmetric monoidal category $(V, \square, 1)$ a V-CATEGORY (or a CATEGORY ENRICHED over V) consists

* a collection of objects,

* for each pair of objects (a_1, a_2) a morphism object $C(a_1, a_2)$ in V
* composition morphisms

$$C(a_2, a_3) \square C(a_1, a_2) \to C(a_1, a_3)$$

PSYCHOLOGICAL CRUTCH

For a V-category C, there is an ordinary category C_0 with the same objects as C and

$$C_0(a, b) := V(1, C(a, b))$$

Example of interest to stable homotopy theory:

$$(V, \square, 1) = (\mathcal{T}, \wedge, S^0)$$

J is the small V-category whose objects are natural # $n \geq 0$. with

$$J(m, n) = \begin{cases} * & \text{if } m > n \\ S^{n-m} & \text{if } m \leq n \end{cases}$$

Composition for $n_1 \leq n_2 \leq n_3$

$$\begin{array}{ccc} J(n_2, n_3) \wedge J(n_1, n_2) & \longrightarrow & J(n_1, n_3) \\ & & \| \\ S^{n_3 - n_2} \wedge S^{n_2 - n_1} & \xrightarrow{\text{iso}} & S^{n_3 - n_1} \end{array}$$

With this definition as a pointed topological category, a J-functor $J \longrightarrow T$ is the same thing as a spectrum.