YONEDA LEMMA (Riehl CTC THM 2.2.4)

LET $\mathcal{C}$ BE A CATEGORY WITH AN OBJECT A. CONSIDER THE FUNCTOR $\text{よ}^A = \mathcal{C}(A, -) : \mathcal{C} \to \mathcal{S}ets$. SUPPOSE $F : \mathcal{C} \to \mathcal{S}et$ IS ANOTHER SUCH FUNCTOR. THEN THE "SET" OF NATURAL TRANSFORMATIONS $\text{よ}^A \Rightarrow F$ IS NATURALLY ISOMORPHIC TO THE SET $F(A)$. THE MAP IS

$$\mathrm{Nat}(\text{よ}^A, F) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} F(A)$$

$$\theta \longmapsto \theta_A(1_A)$$

RECALL A NATURAL TRANSFORMATION

$$A \in \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{D} \end{tikzcd} \qquad \begin{tikzcd} FA \arrow[d, "\theta_A"] \\ GA \end{tikzcd}$$

INCLUDES FOR EACH OBJECT A IN $\mathcal{C}$, A MORPHISM $\theta_A : FA \to GA$ IN $\mathcal{D}$

PROOF WE WILL CONSTRUCT AN INVERSE TO Φ, $\Psi : F(A) \to \mathrm{Nat}(-,-)$

LET $x \in F(A)$. WE NEED TO DEFINE A NATURAL TRANSFORMATION $\Psi x : \text{よ}^A \Rightarrow F$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\text{よ}^A"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow\Psi x"] & \mathcal{S}et \end{tikzcd}$$

LET $A \xrightarrow{f} B \xrightarrow{g} B'$ BE MORPHISMS IN $\mathcal{C}$

OF MORPHISMS IN $\mathcal{C}$

$$\begin{tikzcd}
1_A \in \mathcal{C}(A,A) = h^A(A) \arrow[r, "(\Psi x)_A"] \arrow[d, "f_*"', "h^A(f)"] & F(A) \arrow[d, "Ff"] & 1_A \mapsto (\Psi x)_A(1_A) = x \arrow[d, mapsto] \\
f \in \mathcal{C}(A,B) = h^A(B) \arrow[r, "(\Psi x)_B"] \arrow[d] & F(B) \arrow[d, "Fg"] & f \mapsto (\Psi x)_B(f) = Ff(x) \arrow[d, mapsto] \\
gf \in \mathcal{C}(A,B') = h^A(B') \arrow[r, "(\Psi x)_{B'}"] & F(B') & gf \mapsto (Fg)(Ff)(x)
\end{tikzcd}$$

THUS Ψx IS A NATURAL TRANS.

$h^A \Rightarrow F$ WITH $\Phi(\Psi x) = x$

AS DESIRED.

WE NEED TO SHOW NATURAL IN BOTH F AND A

FOR THE FORMER, SUPPOSE WE HAVE

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{S}et \end{tikzcd}$$. THEN

WE NEED TO ^{G} SHOW

$$\begin{tikzcd}
\mathrm{Nat}(\mathcal{L}^A, F) \arrow[r, "\Phi_F", "\cong"'] \arrow[d, "\beta_*"'] & F(A) \arrow[d, "\beta_A"] \\
\mathrm{Nat}(\mathcal{L}, G) \arrow[r, "\cong", "\Phi_G"'] & G(A)
\end{tikzcd}$$

FOR THE LATTER, GIVEN $f: A \to B$ IN $\mathcal{C}$ WE NEED

$$\begin{tikzcd}
\mathrm{Nat}(\mathcal{L}^A, F) \arrow[r, "\Phi_F", "\cong"'] \arrow[d, "(f^*)^*"'] & F(A) \arrow[d, "Ff"] \\
\mathrm{Nat}(\mathcal{L}^B, F) \arrow[r, "\cong", "\Phi_F"'] & F(B)
\end{tikzcd}$$

SEE RIEHL pp 58-59.

QED

AS NOTED BEFORE, WE GET A FUNCTOR

$$\mathcal{C}^{op} \xrightarrow{\mathcal{L}} Set^{\mathcal{C}} = \text{CATEGORY OF FUNCTORS } \mathcal{C} \longrightarrow Set$$

$$A \longmapsto \mathcal{L}^A = \mathcal{C}(A,-)$$

$$\mathcal{C} \longmapsto Set^{\mathcal{C}^{op}}$$

$$B \longmapsto \mathcal{L}_B = \mathcal{C}(-,B)$$

BOTH FUNCTORS ARE FULL AND FAITHFUL HENCE $\mathcal{L}$ IS CALLED THE YONEDA <u>EMBEDDING</u>.

WE ALSO GET A FUNCTOR

$$\mathcal{C}^{op} \times \mathcal{C} \xrightarrow{\mathcal{C}(-,-)} Set$$

$$(A, B) \longmapsto \mathcal{C}(A, B)$$

THIS IS ALSO FULL + FAITHFUL.

TWO SLOGANS

① A CATEGORY IS DETERMINED BY ITS MORPHISMS

② SUPPOSE WE HAVE TWO YONEDA FUNCTORS $\mathcal{L}^A$ AND $\mathcal{L}^B$. THEN

$$\mathrm{Nat}(\mathcal{L}^A, \mathcal{L}^B) = \mathcal{C}(A, B)$$

A NATURAL TRANSFORMATION BETWEEN REPRESENTABLE FUNCTORS CORRESPONDS TO A MORPHISM BETWEEN THEIR REPRESENTING OBJECTS.

VERY USEFUL

EXAMPLE (RIEHL EX 2.2.2, PROP 2.2.3)

RECALL FOR A GROUP G, WE HAVE A ONE OBJECT CATEGORY BG. CALL THIS OBJECT $[G]$

$$\mathcal{L}^{[G]}[G] = BG([G],[G])$$

$$= \text{THE SET } G$$

SUPPOSE $BG \xrightarrow{X} \mathcal{S}et$ IS ANOTHER SUCH FUNCTOR. IT DEFINES AN ACTION OF G ON THE SET X.

A NATURAL TRANS BETWEEN TWO SUCH FUNCTORS X AND Y A MAP $X \to Y$ WHICH RESPECTS

THE G-ACTIONS ON X AND Y

YONEDA LEMMA SAYS

$$\mathrm{Nat}(\mathcal{L}^{[G]}, X) = X[G]$$

$$= \text{THE SET } X$$

REINTERPRETION:

EACH $x \in X$ DEFINES AN EQUIVARIANT MAP

$$G \longrightarrow X$$
$$\gamma \longmapsto \gamma(x)$$

THE IMAGE OF THIS MAP IS THE ORBIT OF x.

TOMORROW 10 AM TALK BY EMILY RIEHL