# ADJOINT FUNCTORS



RECALL

$$X \in \mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{D} \ni Y \qquad (\perp)$$

WE SAY $F \dashv G$

KAN TURNSTILE

IF THERE IS A NATURAL (IN BOTH X AND Y) $\mathcal{D}(FX, Y) \xrightarrow[\simeq]{\varphi_{X,Y}} \mathcal{C}(X, UY)$

THIS CAN BE INTERPRETED AS A NATURAL EQUIVALENCE BETWEEN TWO FUNCTORS

$$(X,Y) \in \mathcal{C}^{op} \times \mathcal{D} \;\begin{tikzcd} {} \arrow[r, bend left, "\mathcal{D}(F-,-)"] \arrow[r, bend right, "\mathcal{C}(-,U-)"'] \arrow[r, phantom, "\varphi \Downarrow \simeq"] & \mathcal{S}ets \end{tikzcd}$$

$F$ = LEFT ADJOINT

$U$ = RIGHT ADJOINT

SEE NOTES OF 2/10/21 FOR EXAMPLE

⚠ NOT ALL FUNCTORS HAVE LEFT OR RIGHT ADJOINTS

THOSE THAT DO ENJOY SPECIAL PROPERTIES. THERE ARE FUNCTORS THAT HAVE BOTH.

<u>PROP</u> THE COMPOSITE OF TWO LEFT (RIGHT) ADJOINTS IS AGAIN A LEFT (RIGHT) ADJOINT.

PROOF

$$X \in \mathcal{C} \xrightarrow[\varphi' \perp]{F_1} \mathcal{D} \xrightarrow[\varphi'' \perp]{F_2} \mathcal{E} \ni Z$$

$$X \in \mathcal{C} \underset{U_1}{\overset{F_1}{\underset{\varphi' \;\perp}{\rightleftarrows}}} \mathcal{D} \underset{U_2}{\overset{F_2}{\underset{\varphi'' \;\perp}{\rightleftarrows}}} \mathcal{E} \ni Z$$

$$\mathcal{E}(F_2F_1X, Z) \xrightarrow[\cong]{\varphi''_{F_1X,Z}} \mathcal{D}(F_1X, U_2Z) \xrightarrow[\cong]{\varphi'_{X,U_2Z}} \mathcal{C}(X, U_1U_2Z)$$

THIS MEANS $F_2F_1 \dashv U_1U_2$,

SO $F_2F_1$ IS A LEFT ADJOINT AND $U_1U_2$ IS A RIGHT ADJOINT AS CLAIMED. QED

⚠ IT IS A BAD IDEA TO COMPOSE A LEFT AND RIGHT ADJOINT IN EITHER ORDER.

---

SUPPOSE WE HAVE

$$X \qquad \mathcal{C} \underset{U}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathcal{D} \qquad Y$$

$$\mathcal{D}(FX, Y) \xrightarrow[\cong]{\varphi_{X,Y}} \mathcal{C}(X, UY)$$

TWO SPECIAL CASES

① $Y = FX$

$$\mathcal{D}(FX, FX) \xrightarrow{\varphi_{X,FX}} \mathcal{C}(X, UFX)$$

$$1_{FX} \longmapsto X \xrightarrow{\eta_X} UFX$$

$\eta_X$ IS THE UNIT OF THE ADJUNCTIO , IT IS A COMPONENT OF A NATURAL TRANSFORMATION

$$1_{\mathcal{C}} \xRightarrow{\eta} UF$$

EXAMPLE $\mathcal{C} = \mathcal{S}et \quad \mathcal{D} = \mathcal{A}b$

F = FREE ABELIAN GROUP FUNCTOR

U = FORGETFUL FUNCTOR

X IS A SET $\quad \eta_X : X \longrightarrow UFX$

$x \longmapsto$ GENERATOR $[x]$ OF FX.

② $X = UY \rightsquigarrow \quad FUY \xrightarrow{\varepsilon_Y} Y$

COUNIT OF ADJUNCTION, PART OF A NATURAL TRANSFORMATION

$$FU \xRightarrow{\eta} 1_{\mathcal{D}}$$

EXAMPLE : $\mathcal{C}$, $\mathcal{D}$, F AND U AS BEFORE.

Y IS AN ABELIAN GROUP

FUY IS THE FREE ABELIAN GR GENERATED BY THE SET OF ELEMENTS OF Y.

= SET OF FINITE INTEGER LINEAR

= SET OF INTEGER LINEAR COMBINATIONS OF ELEMENTS OF $Y$.

SUCH A LINEAR EXPRESSION DETERMINES AN ELEMENT OF $Y$. THIS DEFINES THE GROUP HOMOMORPHISM

$$FUY \xrightarrow{\varepsilon_Y} Y.$$

---

# LIMITS AND COLIMITS

EXAMPLE: PULLBACK IN $\mathcal{C}$

$$\begin{tikzcd}
X \arrow[dr, dashed, "\lambda"] \arrow[drr, bend left, "h_1"] \arrow[ddr, bend right, "h_2"'] & & \\
& P \arrow[r, "g_1"] \arrow[d, "g_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y_1 \arrow[d, "f_1"] \\
& Y_2 \arrow[r, "f_2"'] & Y_0
\end{tikzcd}$$

PULLBACK SYMBOL

$f_1 h_1 = f_2 h_2$

$f_1 g_1 = f_2 g_2$

$P = \lim_J Y$

$$\mathcal{C}^J(\Delta X, Y) \cong \mathcal{C}(X, P)$$

P SHOULD HAVE A UNIVERSAL PROPERTY: FOR EACH $X$ AS ABOVE, $\exists!\ X \xrightarrow{\lambda} P$ SUCH THAT $h_1 = g_1\lambda$ AND $h_2 = g_2\lambda$

$P$ IS CALLED THE <u>PULLBACK</u> OF THE ORIGINAL DIAGRAM

$Y \qquad Y_1 \xrightarrow{f_1} Y_0 \xleftarrow{f_2} Y_2$

IN ...

IN $\mathcal{S}et$,

$$P = \{(y_1, y_2) \in Y_1 \times Y_2 : f_1(y_1) = f_2(y_2) \in Y_0\}$$

THIS IS A TYPE OF LIMIT

ANOTHER EXAMPLE: REVERSE ALL ARROWS. THE RESULTING OBJECT IS A PUSHOUT, A TYPE OF COLIMIT.

REINTERPRETATION

LET $J$ DENOTE THE CATEGORY

$$1 \longrightarrow 0 \longleftarrow 2$$

OUR DIAGRAM $Y$ IS A FUNCTOR

$$J \longrightarrow \mathcal{C}$$

$$0 \longmapsto Y_0$$

$$1 \longmapsto Y_1$$

$$2 \longmapsto Y_2$$

LET $\mathcal{C}^J$ DENOTE THE CATEGORY OF ALL SUCH FUNCTOR

SUPPOSE $Y'$ IS ANOTHER SUCH DIAGRAM. A MORPHISM IN $\mathcal{C}^J$ $\quad Y \xrightarrow{\ell} Y'$

IS

$$\begin{tikzcd} Y_1 \arrow[r, "f_1"] \arrow[d, "\ell_1"'] & Y_0 \arrow[d, "\ell_0"] & Y_2 \arrow[l, "f_2"'] \arrow[d, "\ell_2"] \\ Y_1' \arrow[r, "f_1'"] & Y_0' & Y_2' \arrow[l, "f_2'"'] \end{tikzcd}$$

THERE IS A <u>DIAGONAL FUNCTOR</u> $\mathcal{C} \xrightarrow{\Delta} \mathcal{C}^J$

$$X \qquad\qquad \begin{matrix} X \\ \downarrow \\ X \\ \uparrow \\ X \end{matrix}$$

CONSTANT $X$-VALUED DIAGRAM

SUPPOSE PULLBACKS EXIST IN $\mathcal{C}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left=2, "\Delta"] \arrow[r, phantom, "\perp"] & \mathcal{C}^J \arrow[l, shift left=2, "\lim_J"] \end{tikzcd}$$

$\lim_J Y$ $=$ PULLBACK OBJECT

EXERCISE: $\lim_J$ IS THE RIGHT ADJOINT OF $\Delta$.

PUSHOUT DISCUSSION: REPLACE $J$ BY $J^{op}$

$$1 \longleftarrow 0 \longrightarrow 2$$

$\operatorname{colim}_{J^{op}} Y$ = PUSHOUT

AND

$$\begin{tikzcd} C \arrow[r, "\Delta"'] \arrow[r, phantom, shift left=3, "\perp"] & C^J \arrow[l, bend right, "\operatorname{colim}_J"'] \end{tikzcd}$$

IF $C$ HAS PUSHOUTS, THEY ARE RELATED TO THE DIAGONAL FUNCTOR AS SHOWN.

EXERCISE:

FOR $J$ THE PULLBACK CATEGORY

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"'] & C^J \arrow[l, bend right, "\text{BORING}"'] \arrow[l, bend left, "\underset{\text{PULLBACK}}{\lim_J}"] \end{tikzcd}$$

BORING$(Y) = Y_0$

## GENERIZATION

WE COULD REPLACE $J$ BY <u>ANY</u> SMALL CATEGORY

WE STILL HAVE $\mathcal{C} \xrightarrow{\Delta} \mathcal{C}^J$

$\mathcal{C}^J$ = "CATEGORY OF $J$-SHAPED DIAGRAMS IN $\mathcal{C}$"

WE CAN ASK FOR LEFT AND RIGHT ADJOINTS.

<u>DEF</u> $\mathcal{C}$ IS <u>(CO)COMPLETE</u> IF ALL SMALL (CO)LIMITS EXIST, AND <u>BICOMPLETE</u> IF BOTH ARE TRUE.

$\mathcal{S}et$, $\mathcal{T}op$, $\mathcal{A}b$, ETC. ARE BICOMPLETE.

<u>EXAMPLES</u> $\mathcal{C}$ BICOMPLETE

(1) $J$ = EMPTY CATEGORY

(NO OBJECTS)

ONLY ONE FUNCTOR, THE EMPTY DIAGRAM

ITS LIMIT / COLIMIT

EXERCISE: ARE TERMINAL / INITIAL OBJECTS IN $\mathcal{C}$.

(2) $J$ IS DISCRETE, I E ITS ONLY MORPHISMS ARE IDENTITIES.

A FUNCTOR $J \longrightarrow \mathcal{C}$

IS A SET (OF THE RIGHT SIZE) OF OBJECTS IN $\mathcal{C}$.

(CO)LIMIT = (CO)PRODUCT OF THESE OBJECTS

$$\prod_{j \in J} X_j \qquad \coprod_{j \in J} X_j$$

CARTESIAN PRODUCT      DISJOINT UNION

CARTESIAN
PRODUCT
IN *Set*

DISJOINT
UNION
IN *Set*