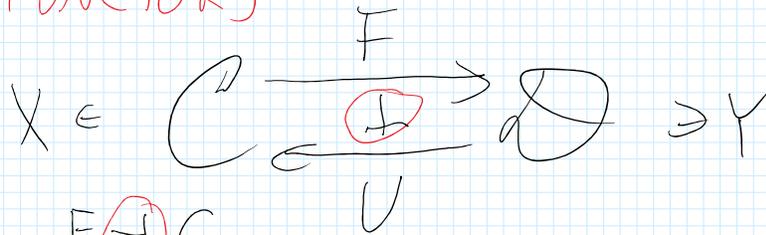


ADJOINT FUNCTORS

Tuesday, February 16, 2021 7:55 AM

RECALL



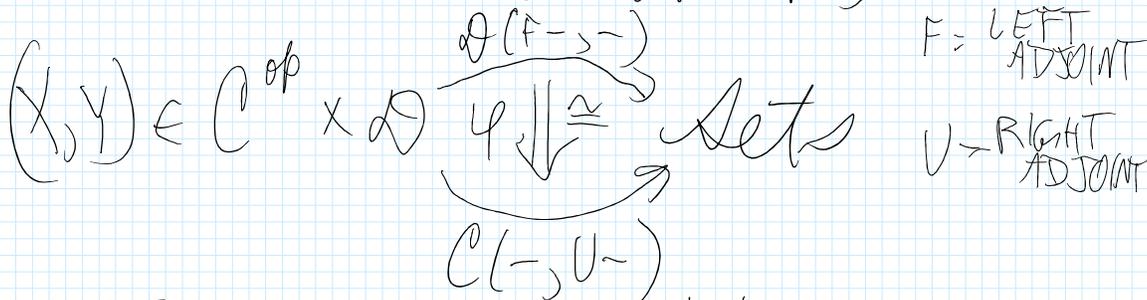
WE SAY $F \dashv U$

KAN TURNSTILE

IF THERE IS A NATURAL (IN BOTH X AND Y)

$$\mathcal{D}(F X, Y) \xrightarrow[\cong]{\eta_{X,Y}} \mathcal{C}(X, U Y)$$

THIS CAN BE INTERPRETED AS A NATURAL EQUIVALENCE BETWEEN TWO FUNCTORS



SEE NOTES OF 2/10/21 FOR EXAMPLE



NOT ALL FUNCTORS HAVE LEFT OR RIGHT ADJOINTS THOSE THAT DO ENJOY SPECIAL PROPERTIES. THERE ARE FUNCTORS THAT HAVE BOTH.

PROP THE COMPOSITE OF TWO LEFT (RIGHT) ADJOINTS IS AGAIN A LEFT (RIGHT) ADJOINT.

PROOF =



$$\begin{array}{c}
 X \in \mathcal{C} \xrightarrow{\varphi''} \mathcal{D} \xrightarrow{\varphi''} \mathcal{E} \ni Z \\
 \leftarrow U_1 \quad \leftarrow U_2 \\
 \mathcal{C}(F_2 F_1 X, Z) \xrightarrow[\cong]{\varphi''_{F_1 X, Z}} \mathcal{D}(F_1 X, U_2 Z) \\
 \xrightarrow[\cong]{\varphi'_{X, U_2 Z}} \mathcal{C}(X, U_1 U_2 Z)
 \end{array}$$

THIS MEANS $F_2 F_1 \dashv U_1 U_2$,
 SO $F_2 F_1$ IS A LEFT ADJOINT
 AND $U_1 U_2$ IS A RIGHT ADJOINT
 AS CLAIMED. QED

 IT IS A BAD IDEA TO
 COMPOSE A LEFT AND
 RIGHT ADJOINT IN EITHER
 ORDER.

SUPPOSE WE HAVE

$$X \quad \mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \perp \\ \xleftarrow{U} \end{array} \mathcal{D} \quad Y$$

$$\mathcal{D}(FX, Y) \xrightarrow[\cong]{\varphi_{X, Y}} \mathcal{C}(X, UY)$$

TWO SPECIAL CASES

① $Y = FX$

$$\mathcal{D}(FX, FX) \xrightarrow{\varphi_{X, FX}} \mathcal{C}(X, UFX)$$

$$1_{FX} \longmapsto X \xrightarrow{\eta_X} UFX$$

η_X IS THE UNIT OF THE
ADJUNCTION, η IS A
COMPONENT OF A NATURAL
TRANSFORMATION

$$1_C \xrightarrow{\eta} UF$$

EXAMPLE $C = \text{Set}$ $D = \text{Ab}$

$F = \text{FREE ABELIAN GROUP FUNCTOR}$

$U = \text{FORGETFUL FUNCTOR}$

X IS A SET $\eta_X: X \rightarrow UFX$

$x \mapsto \text{GENERATOR}$
 $[x]$ OF FX .

$$\textcircled{2} X = UY \rightsquigarrow FUY \xrightarrow{\epsilon_Y} Y$$

COUNIT OF ADJUNCTION,
PART OF A NATURAL TRANSFORMATION

$$FU \xrightarrow{\eta} 1_D$$

EXAMPLE: C, D, F AND U
AS BEFORE.

Y IS AN ABELIAN GROUP

FUY IS THE FREE ABELIAN GR
GENERATED BY THE SET
OF ELEMENTS OF Y .

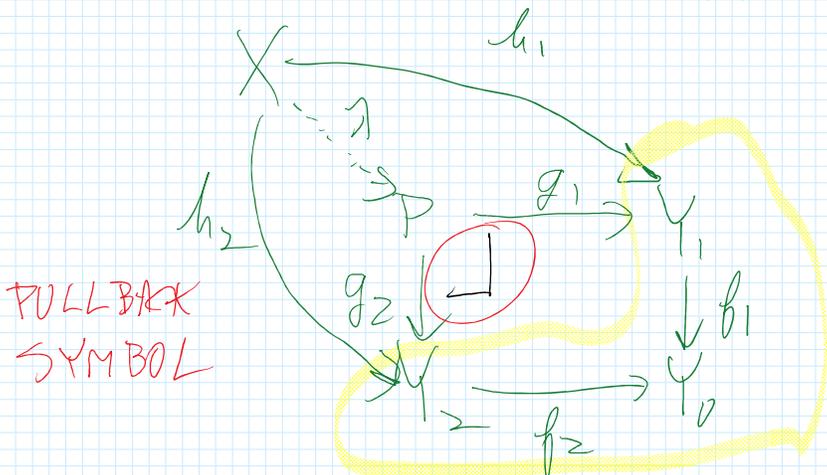
= SET OF ^{FINITE} INTEGER LINEAR

= SET OF ALL INTEGER LINEAR COMBINATIONS OF ELEMENTS OF Y .

SUCH A LINEAR EXPRESSION DETERMINES AN ELEMENT OF Y . THIS DEFINES THE GROUP HOMOMORPHISM $f: Y \rightarrow Y$.

LIMITS AND COLIMITS

EXAMPLE: PULLBACK IN \mathcal{C}



$$f_1 h_1 = f_2 h_2$$

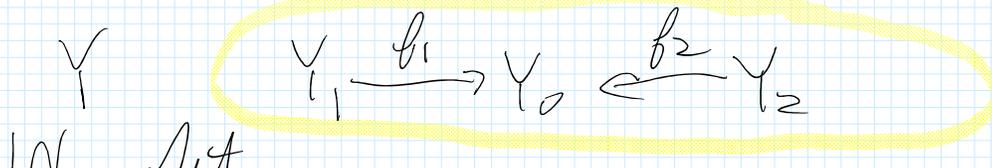
$$f_1 g_1 = f_2 g_2$$

$$P = \lim_{\leftarrow} Y$$

$$\mathcal{C}(\Delta X, Y)$$

$$\mathcal{C}(X, P)$$

P SHOULD HAVE A UNIVERSAL PROPERTY: FOR EACH X AS ABOVE, $\exists!$ $X \xrightarrow{\lambda} P$ SUCH THAT $h_1 = g_1 \lambda$ AND $h_2 = g_2 \lambda$. P IS CALLED THE PULLBACK OF THE ORIGINAL DIAGRAM



IN Set ,

$$P = \left\{ (Y_1, Y_2) \in Y_1 \times Y_2 : f_1(Y_1) = f_2(Y_2) \in Y_0 \right\}$$

THIS IS A TYPE OF LIMIT

ANOTHER EXAMPLE: REVERSE
ALL ARROWS. THE RESULTING
OBJECT IS A PUSHOUT,
A TYPE OF COLIMIT.

REINTERPRETATION

LET J DENOTE THE CATEGORY

$$1 \longrightarrow 0 \longleftarrow 2$$

OUR DIAGRAM Y IS A FUNCTOR

$$J \longrightarrow \mathcal{C}$$

$$0 \longmapsto Y_0$$

$$1 \longmapsto Y_1$$

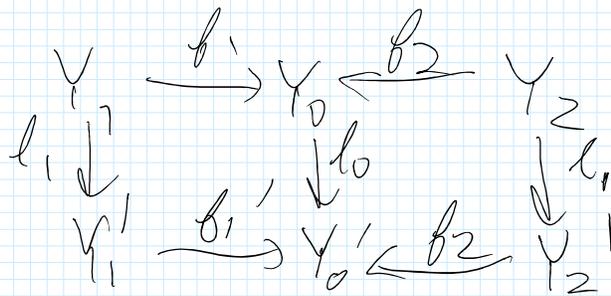
$$2 \longmapsto Y_2$$

LET \mathcal{C}^J DENOTE THE
CATEGORY OF ALL SUCH
FUNCTOR

SUPPOSE Y' IS ANOTHER
SUCH DIAGRAM. A MORPHISM

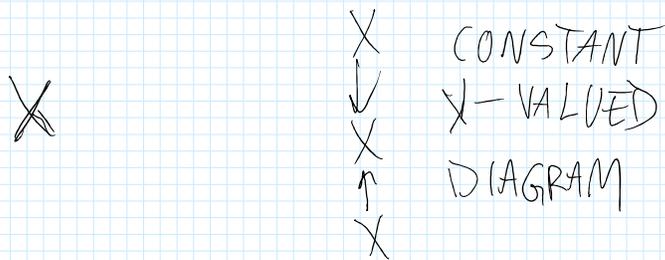
$$\text{IN } \mathcal{C}^J \quad Y \xrightarrow{\ell} Y'$$

IS

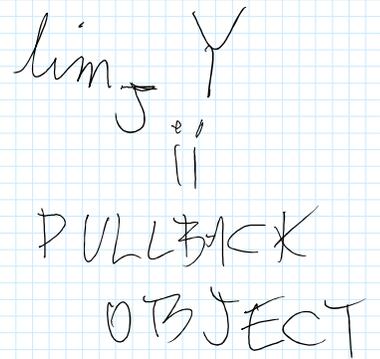
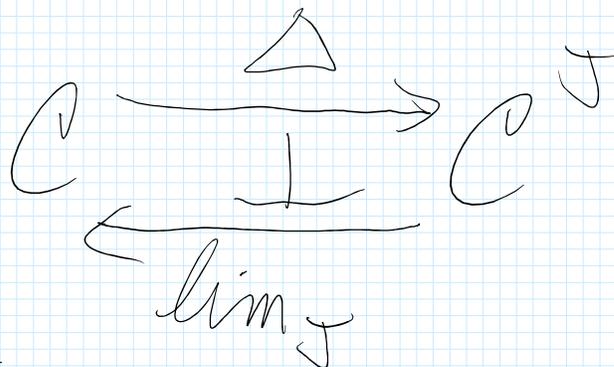


THERE IS A DIAGONAL
FUNCTOR

$$\mathcal{C} \xrightarrow{\Delta} \mathcal{C}^J$$



SUPPOSE PULLBACKS EXIST IN \mathcal{C}



EXERCISE:

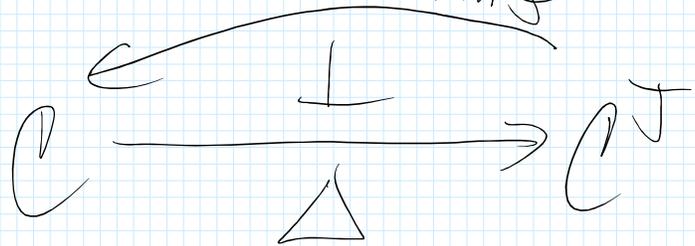
\lim_J IS THE RIGHT ADJOINT
OF Δ .

PUSHOUT DISCUSSION:
REPLACE J BY J^{op}

$$1 \leftarrow 0 \rightarrow 2$$

$\text{colim}_{j \in J} Y = \text{PUSHOUT}$

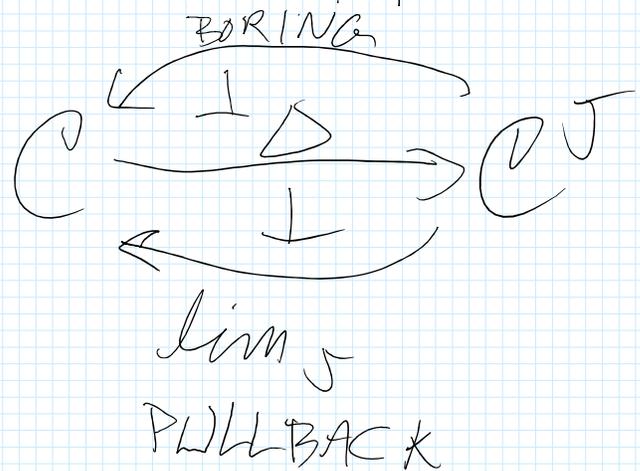
AND



IF C HAS PUSHOUTS
THEY ARE RELATED TO
THE DIAGONAL FUNCTOR
AS SHOWN.

EXERCISE:

FOR J THE PULLBACK
CATEGORY



$$\text{BORING}(Y) = Y_0$$

GENERALIZATION

WE COULD REPLACE J
BY ANY SMALL CATEGORY

WE STILL HAVE $\mathcal{C} \xrightarrow{\Delta} \mathcal{C}^J$

$\mathcal{C}^J =$ "CATEGORY OF J -SHAPED
DIAGRAMS IN \mathcal{C} "

WE CAN ASK FOR LEFT AND
RIGHT ADJOINTS.

DEF \mathcal{C} IS (CO)COMPLETE

IF ALL ^{SMALL} (CO)LIMITS EXIST,

AND BICOMPLETE IF

BOTH ARE TRUE.

Set, Top, Ab, ETC. ARE
BICOMPLETE.

EXAMPLES \mathcal{C} BICOMPLETE

(1) $J =$ EMPTY CATEGORY

(NO OBJECTS)
ONLY ONE FUNCTOR, THE
EMPTY DIAGRAM

ITS LIMIT / COLIMIT

EXERCISE: ARE TERMINAL / INITIAL
OBJECTS IN \mathcal{C} ,

② J IS DISCRETE, I.E.
ITS ONLY MORPHISMS ARE
IDENTITIES.

A FUNCTOR $J \rightarrow \mathcal{C}$

IS A SET (OF THE RIGHT SIZE)
OF OBJECTS IN \mathcal{C} .

(CO) LIMIT = (CO) PRODUCT OF
THESE OBJECTS

$\prod_{j \in J} X_j$

CARTESIAN
PRODUCT

$\coprod_{j \in J} X_j$

DISJOINT
UNION

CARTESIAN
PRODUCT
IN Set

DISJOINT
UNION
IN Set