Wednesday, February 24, 2021 1:54 PM

POSSIBLE FUTURE TOPICS
SYMMETRIC MONOIDAL CATEGORIES
ENRICHED CATEGORIES
SIMPLICIAL SETS (NEEDED FOR $\infty$-CATEGORIES
MODEL CATEGORIES, USED IN HOMOTOPY THEORY,
SUGGEST OTHERS!

TODAY'S TOPIC: KAN EXTENSIONS
REFERENCES: RIEHL CTC CHAPTER 6, HHR 2,5.

SEE PAGE 190 OF CTC FOR THE DEF OF KAN EXTENSIONS

LEFT (RIGHT) KAN EXTENSIONS ARE RELATED TO COLIMITS (LIMITS).

EXAMPLE 6.1.3

$$\begin{tikzcd} \mathbb{1} \arrow[r, "*"] \arrow[dr, "A"'] \arrow[r, phantom, shift right=4, "\Downarrow x"] & \mathcal{S}et \\ & \mathcal{C} \arrow[u, dashed, "F"'] \end{tikzcd}$$

THERE IS A NAT TRANS FOR EACH $x \in FA$

$\mathbb{1}$ DENOTES THE TRIVIAL CATEGORY. IT HAS ONE OBJECT AND ONE MORPHISM.

$*$ CHOOSES THE ONE ELEMENT SET

$A$ IS AN OBJECT IN $\mathcal{C}$

THE "BEST" CHOICE OF $F$, THE LEFT KAN EXTENSION, $Lan_A * = \mathcal{C}(A,-) = \mathcal{L}^A$

YONEDA LEMMA IDENTIFIES $Nat(\mathcal{L}^A, F)$ WITH $F(A)$

EXAMPLE 6.1.4 $G$ = GROUP

$BG$ = ONE OBJECT CATEGORY AS BEFORE

$$\begin{tikzcd} \mathbb{1} \arrow[r, "A"] \arrow[d] & \mathcal{C} \\ BG \arrow[ur, dashed, "X"'] \end{tikzcd}$$

$A$ IS AN OBJECT IN $\mathcal{C}$

$\exists!$ FUNCTOR $\mathbb{1} \longrightarrow BG$

$X$ IS AN OBJECT IN $\mathcal{C}$ EQUIPPED WITH A $G$-ACTION

IF $\mathcal{C}$ HAS COPRODUCTS, THEN THE LEFT KAN EXTENSION IS

$\coprod_G A$ = "DISJOINT UNION OF #G COPIES OF A" PERMUTED BY G.

IF $\mathcal{C}$ HAS PRODUCTS, THEN THE RIGHT KAN EXTENSION IS

$\prod_G A$ = "CARTESIAN PRODUCT" WITH CO-ORDINATES PERMUTED BY G.

ALTERNATE INTERPRETATION OF KAN EXTENSIONS

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & & \mathcal{E} \\ & \mathcal{D} \arrow[ur, dotted] & \end{tikzcd}$$

LET $\mathcal{E}^{\mathcal{C}}$ = CATEGORY OF FUNCTORS $\mathcal{C} \to \mathcal{E}$

$\mathcal{E}^{\mathcal{D}}$ = " $\mathcal{D} \to \mathcal{E}$

K INDUCES

$$\begin{tikzcd} \mathcal{E}^{\mathcal{D}} \arrow[rr, "K^*"] & & \mathcal{E}^{\mathcal{C}} \arrow[ll, bend right, "\mathrm{Lan}_K"', "\perp"] \arrow[ll, bend left, "\mathrm{Ran}_K", "\perp"'] \end{tikzcd}$$

WE ARE LOOKING FOR LEFT AND RIGHT ADJOINTS OF $K^*$

THIS IS THE SUBJECT OF CTC PROP 6.1.5

RECALL THAT ADJUNCTIONS LEAD TO UNITS $\eta$ AND COUNITS $\varepsilon$

KAN'S $\eta$ IS THE UNIT OF THE ADJUNCTION $\mathrm{Lan}_K \dashv K^*$

HIS $\varepsilon$ IS THE COUNIT OF $K^* \dashv \mathrm{Ran}_K$.

EXAMPLE 6.1.7

$k$ = FIELD $\quad \mathrm{Vect}_k$ = CATEGORY OF $k$-VECTOR SPACES

$G$ = GROUP $\quad BG$ IS AS BEFORE

$G \geq H$ = SUBGROUP $\quad BH \subset BG$. FINITE INDEX ?

$\mathrm{Vect}_k^{BG}$ = CATEGORY OF FUNCTORS $BG \longrightarrow \mathrm{Vect}_k$

= CAT OF REPRESENTATIONS OF $G$ OVER $k$.

THERE IS A RESTRICTION FUNCTOR

$$\mathrm{res}^G_H : \mathrm{Vect}_k^{BG} \longrightarrow \mathrm{Vect}_k^{BH}$$

$$\mathrm{res}^G_H : \mathrm{Vect}_k^{BG} \longrightarrow \mathrm{Vect}_k^{BH}$$

$$\begin{tikzcd}
\mathrm{Vect}_k^{BG} \arrow[rr, "\mathrm{res}^G_H"'] & & \mathrm{Vect}_k^{BH} \arrow[ll, bend right=40, "\mathrm{ind}^G_H"'] \arrow[ll, bend left=40, "\mathrm{coind}^G_H"] \arrow[ll, phantom, shift right=3ex, "\perp"] \arrow[ll, phantom, shift left=3ex, "\perp"]
\end{tikzcd}$$

$$BH \xrightarrow{\ \kappa\ } BG \qquad \mathrm{Vect}_k^{BH} \xleftarrow[\mathrm{res}^G_H]{\kappa^*} \mathrm{Vect}_k^{BG}$$

$\mathrm{ind}^G_H$ AND $\mathrm{coind}^G_H$ ARE LEFT AND RIGHT KAN EXTENSIONS ALONG $\kappa$

<u>EXAMPLE</u> AS IN 6.1.7, REPLACE $\mathrm{Vect}_k$ BY $\mathcal{S}et$

$$\begin{tikzcd}
\mathcal{S}et^{BG} \arrow[rr, "\mathrm{res}^G_H"'] & & \mathcal{S}et^{BH} \arrow[ll, bend right=40, "G\times_H(-)"'] \arrow[ll, phantom, shift right=3ex, "\perp"]
\end{tikzcd}$$

$\perp$

$\mathrm{Hom}_H(G, -)$

FOR A $H$-SET $X$

$$G \times_H X = G \times X \,/\, (\gamma h, x) = (\gamma, hx)$$

FOR $\gamma \in G$, $x \in X$, $h \in H$.

= DISJOINT UNION OF $\# G/H$ COPIES OF $X$

$G$ ACTS BY LEFT MULTIPLICATION

THE RIGHT ADJOINT SENDS THE $H$-SET $X$ TO

$\mathrm{Hom}_H(G, X)$ = CARTESIAN PRODUCT OF $\# G/H$ COPIES $X$

= SET OF $H$-EQUIVARIANT MAPS $G \longrightarrow X$,

WHERE $H$ ACTS ON $G$ BY LEFT MULTIPLICATION

$G$ ACTS HERE BY RIGHT MULTIPLICATION)

G ACTS HERE BY RIGHT MULTIPLICATION

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# A FORMULA FOR LEFT/RIGHT Kan EXTENSIONS

SMALL $A \in \mathcal{C}$, $D \in \mathcal{D}$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[dr, "K"'] & \mathcal{E} \\
& \mathcal{D} \arrow[u, dotted]
\end{tikzcd}$$

DEF THE <u>COMMA CATEGORY</u> $K \downarrow D$ HAS AS OBJECTS ARE MORPHISMS $KA \xrightarrow{\alpha} D$

IT MORPHISMS ARE DIAGRAMS INDUCED BY MORPHISMS $A \xrightarrow{h} A'$ IN $\mathcal{C}$

$$\begin{tikzcd}
KA \arrow[rr, "Kh"] \arrow[dr, "\alpha"'] & & KA' \arrow[dl, "\alpha'"] \\
& D &
\end{tikzcd}$$

$K \downarrow D$ IS A SMALL CATEGORY

THERE IS A FUNCTOR

$$K \downarrow D \xrightarrow{\pi^D} \mathcal{C} \xrightarrow{F} \mathcal{E}$$

$$(KA \xrightarrow{\alpha} D) \longmapsto A$$

SUPPOSE THIS COMPOSITE FUNCTOR HAS A COLIMIT

THEN WE CAN DEFINE

$$\mathrm{Lan}_K F\,(D) = \operatorname*{colim}_{K \downarrow D} F\pi^D$$

SIMILARLY FOR RIGHT KAN EXTENSION

SEE THEOREM 6.2.1 OF CK.

COR 6.2.6

SMALL

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & & \mathcal{E} \\
& \mathcal{D} \arrow[ur, dotted] &
\end{tikzcd}$$

a) IF $\mathcal{E}$ IS COCOMPLETE (HAS ALL COLIMITS) THEN THE LEFT KAN EXTENSION EXISTS AND IS DEFINED BY THE COLIMIT ABOVE

b) SIMILARLY FOR RIGHT KAN EXTENSIONS