

POSSIBLE FUTURE TOPICS

SYMMETRIC MONOIDAL CATEGORIES

ENRICHED CATEGORIES

SIMPLICIAL SETS (NEEDED FOR

∞ -CATEGORIES

MODEL CATEGORIES, USED IN
HOMOTOPY THEORY)

SUGGEST OTHERS!

TODAY'S TOPIC: KAN EXTENSIONS

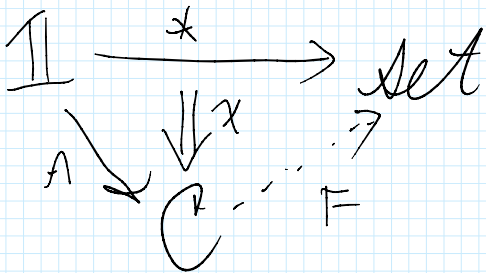
REFERENCES: RIEHL CTC CHAPTER 6,
HHR 2.5.

SEE PAGE 190 OF CTC FOR
THE DEF OF KAN EXTENSIONS

LEFT (RIGHT) KAN EXTENSIONS

ARE RELATED TO COLIMITS (LIMITS).

EXAMPLE 6.1.3



\Downarrow DENOTES THE TRIVIAL CATEGORY. IT HAS ONE OBJECT AND ONE MORPHISM!

x CHOOSES THE ONE ELEMENT SET

A IS AN OBJECT IN \mathcal{C}

THERE IS A NAT TRANS FOR EACH $x \in FA$

THE "BEST" CHOICE OF F , THE LEFT KAN EXTENSION, $\text{Lan}_A x = \mathcal{C}(A, -) = \mathcal{L}^A$

YONEDA LEMMA IDENTIFIES

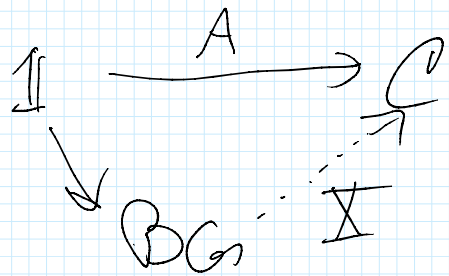
$\text{Nat}(\mathcal{L}^A, F)$ WITH $F(A)$

EXAMPLE 6.1.4

$G = \text{GROUP}$

$\mathcal{B}G = \text{ONE OBJECT CATEGORY AS BEFORE}$

A IS AN OBJECT IN \mathcal{C}



$\exists!$ FUNCTOR $I \rightarrow \mathcal{B}G$

X IS AN OBJECT IN \mathcal{C} EQUIPPED WITH A G -ACTION

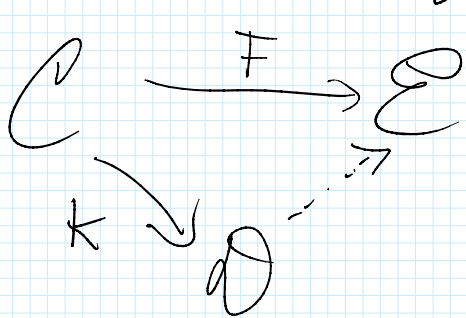
IF \mathcal{C} HAS COPRODUCTS, THEN THE LEFT KAN EXTENSION IS

$\coprod_G A =$ "DISJOINT UNION OF #G COPIES OF A" PERMUTED BY G.

IF \mathcal{C} HAS PRODUCTS, THEN THE RIGHT KAN EXTENSION IS

$\prod_G A =$ "CARTESIAN PRODUCT" WITH CO-ORDINATES PERMUTED BY G.

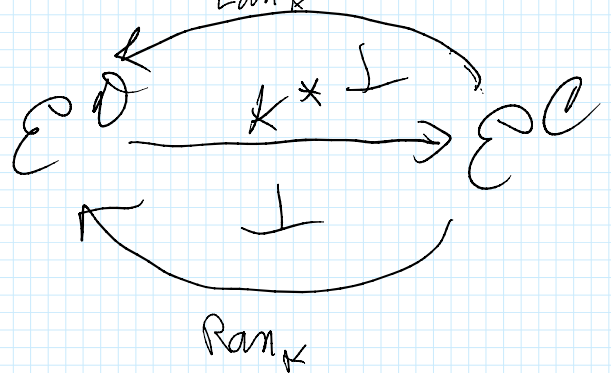
ALTERNATE INTERPRETATION OF KAN EXTENSIONS



LET $\mathcal{E}^{\mathcal{C}} =$ CATEGORY OF FUNCTORS $\mathcal{C} \rightarrow \mathcal{E}$

$\mathcal{E}^{\mathcal{D}} =$ " " $\mathcal{D} \rightarrow \mathcal{E}$

K INDUCES Lan_K



WE ARE LOOKING FOR LEFT AND RIGHT ADJOINTS OF K^*

THIS IS THE SUBJECT OF CTC PROP 6.1.5

RECALL THAT ADJUNCTIONS
LEAD TO UNITS η AND
COUNITS ε

KAN'S η IS THE UNIT OF
THE ADJUNCTION $\text{Lan}_K \dashv K^*$

HIS ε IS THE COUNIT OF
 $K^* \dashv \text{Ran}_K$.

EXAMPLE 6.1.7

$K = \text{FIELD}$ $\text{Vect}_K = \text{CATEGORY OF } K\text{-VECTOR SPACES}$

$G = \text{GROUP}$ BG IS AS BEFORE

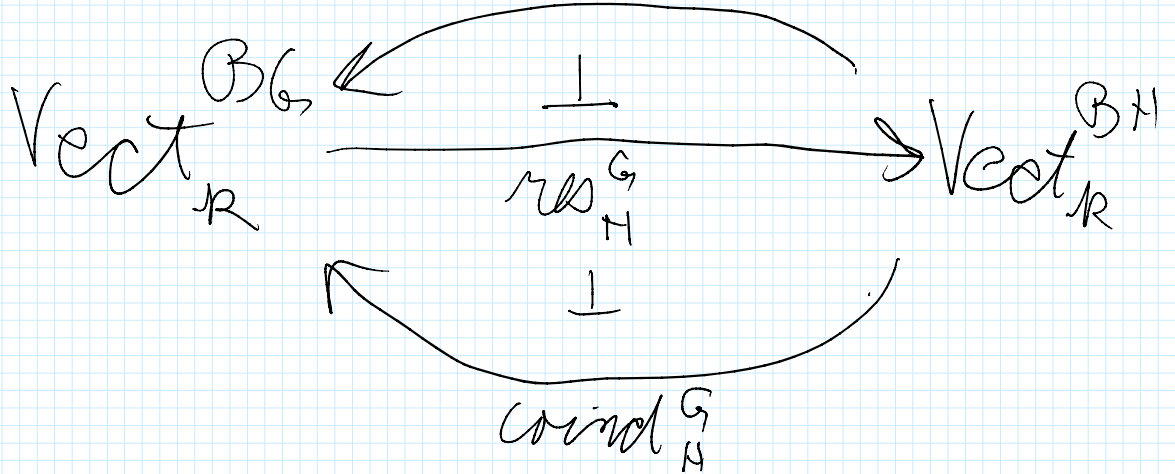
$G \supseteq H = \text{SUBGROUP}$ $\text{BH} \subset \text{BG}$. FINITE INDEX?

$\text{Vect}_K^{\text{BG}} = \text{CATEGORY OF FUNCTORS}$
 $\text{BG} \rightarrow \text{Vect}_K$
 $= \text{CAT OF REPRESENTATIONS}$
 $\text{OF } G \text{ OVER } K.$

THERE IS A RESTRICTION FUNCTOR

$$\text{res}_H^G : \text{Vect}_K^{\text{BG}} \longrightarrow \text{Vect}_K^{\text{BH}}$$

$$\text{res}_H^G: \text{Vect}_R^G \longrightarrow \text{Vect}_R^{BH}$$

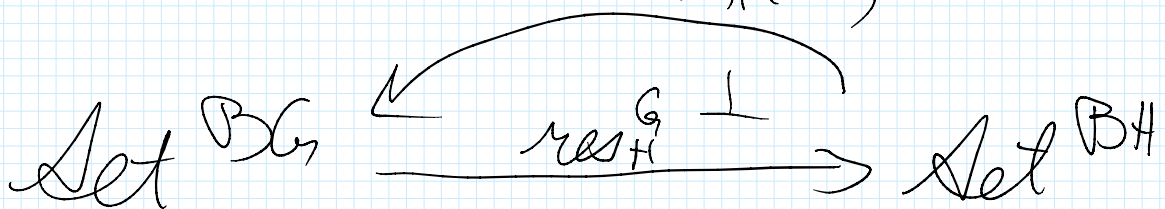


$$BH \xrightarrow{K} BG$$

$$\text{Vect}_R^{BH} \xleftarrow{K^*} \text{Vect}_R^{BG}$$

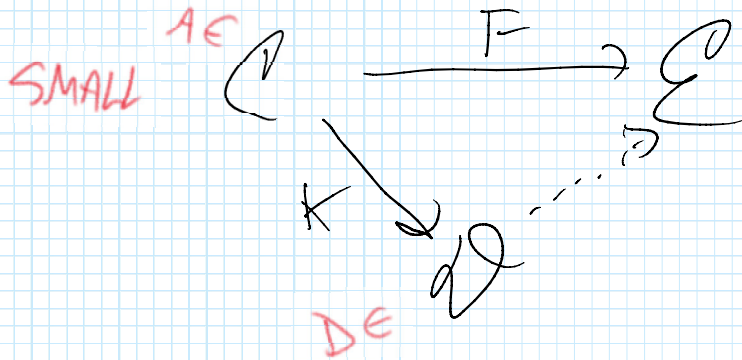
ind_H^G AND coind_H^G ARE LEFT
AND RIGHT KAN EXTENSIONS
ALONG K

EXAMPLE AS IN 6.1.7,
REPLACE Vect_R BY $\text{Set}_{G \times_H (-)}$



ON THIS HERE BY RIGHT
MULTIPLICATION

A FORMULA FOR LEFT / RIGHT Kan EXTENSIONS

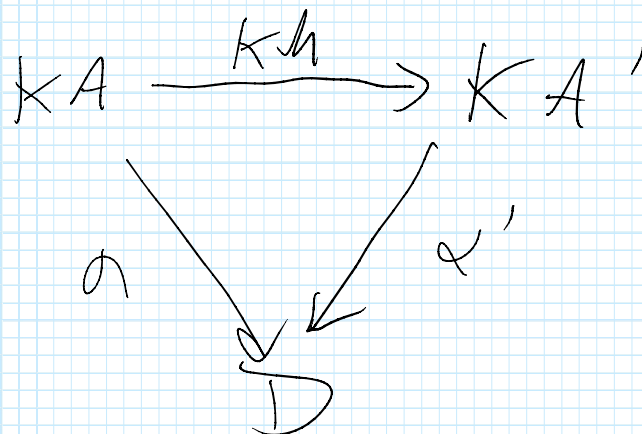


DEF THE COMMA CATEGORY $K \downarrow D$

HAS AS OBJECTS ARE
MORPHISMS $KA \xrightarrow{\alpha} D$

IT MORPHISMS ARE DIAGRAMS
INDUCED BY MORPHISMS

$$A \xrightarrow{h} A' \quad \text{IN } \mathcal{C}$$



$K \downarrow D$ IS A SMALL CATEGORY
 THERE IS A FUNCTOR

$$K \downarrow D \xrightarrow{\pi^D} \mathcal{C} \xrightarrow{F} \mathcal{E}$$

$$(KA \xrightarrow{\alpha} D) \longmapsto A$$

SUPPOSE THIS COMPOSITE
 FUNCTOR HAS A COLIMIT
 THEN WE CAN DEFINE

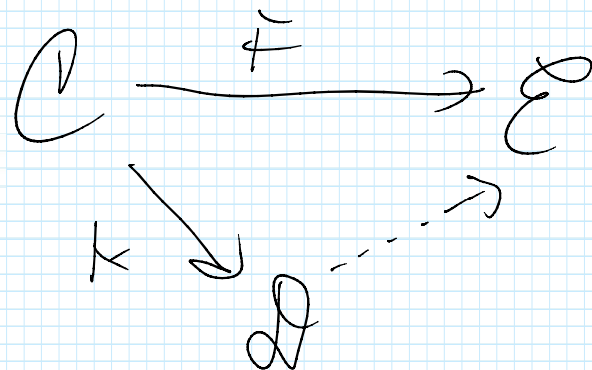
$$\text{Lan}_K F(D) = \text{colim}_{K \downarrow D} F \pi^D$$

SIMILARLY FOR RIGHT KAN
 EXTENSION

SEE THEOREM 6.2.1 OF CR.

COR 6.2.6

SMALL



- a) IF Σ IS COCOMPLETE (HAS ALL COLIMITS) THEN THE LEFT KAN EXTENSION EXISTS AND IS DEFINED BY THE COLIMIT ABOVE
- b) SIMILARLY FOR RIGHT KAN EXTENSIONS