

MORE ABOUT FUNCTORS

RECALL A FUNCTOR



ASSIGNS TO EACH OBJECT $X \in \mathcal{C}$

AN OBJECT FX IN \mathcal{D}

AND TO EACH MORPHISM $X \xrightarrow{g} Y$

IN \mathcal{C} , A MORPHISM $Fg: FX \rightarrow FY$
 IN \mathcal{D}



IT PRESERVES MORPHISM COMPOSITION AND IDENTITY MORPHISMS

EXAMPLES

- 1) LET G BE A GROUP
 \mathcal{B}_G IS THE ONE OBJECT WITH

A MORPHISM FOR EACH $\gamma \in G$.
 A FUNCTOR $BG \xrightarrow{F} \mathcal{C}$
 DETERMINES AN OBJECT $X \in \mathcal{C}$
 AND A COLLECTION OF
 AUTOMORPHISMS OF X RELATED
 TO G . THIS DEFINES AN ACTION
 OF G ON X .

2) LET X BE AN OBJECT IN
 A CATEGORY \mathcal{C} . CONSIDER

$$\mathcal{C} \xrightarrow{\mathcal{C}(X, -)} \text{Set}$$

$$Y \longmapsto \mathcal{C}(X, Y) = \text{SET OF MORPHISMS } X \rightarrow Y \text{ IN } \mathcal{C}.$$

(FUNCTOR COREPRESENTED BY X)

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & & \downarrow g \\ & & Z \end{array}$$

$$\begin{array}{ccc} \mathcal{C}(X, Y) & & \\ \downarrow g_* & & \\ \mathcal{C}(X, Z) & & \end{array}$$

$$\begin{array}{ccc} f: X \rightarrow Y & & \\ \downarrow & & \\ gf: X \rightarrow Z & & \end{array}$$

$$\tilde{Z} \quad C(Y, Z) \quad gf: X \rightarrow Z$$

3) CONSIDER

$$C \xrightarrow{C(-, X)} \text{Set}$$

$$W \longmapsto C(W, X)$$

FUNCTOR REPRESENTED BY X

$$\begin{array}{ccccc}
 W & & C(W, X) & \xrightarrow{gf} & \\
 g \downarrow & & \uparrow & \downarrow & \\
 Z & & C(Z, X) & \xrightarrow{gf} & \\
 & & & & \downarrow \\
 & & & & Z \xrightarrow{g} X
 \end{array}$$

THIS "FUNCTOR" IS CONTRAVARIANT,
 I.E. ARROW REVERSING,

IT IS A FUNCTOR

$$C^{op} \xrightarrow{C(-, X)} \text{Set}$$

$$W \xrightarrow{g} Z$$

THIS FUNCTOR PRESERVES

Σ DIRECTION OF ARROWS.

WE CAN COMBINE THESE LAST TWO INTO A SINGLE FUNCTOR.

DEFINITION LET \mathcal{C} AND \mathcal{D}

BE CATEGORIES. THEIR PRODUCT CATEGORY $\mathcal{C} \times \mathcal{D}$

HAS AS OBJECTS PAIRS (X, Y)

WITH $X \in \mathcal{C}$ AND $Y \in \mathcal{D}$

AND MORPHISMS $(X, Y) \rightarrow (X', Y')$

ARE PAIRS (f, g) WHERE

$f: X \rightarrow X'$ IS A MORPHISM IN \mathcal{C}

$g: Y \rightarrow Y'$ " " \mathcal{D}

COMPOSITION IS OBVIOUS.

4) WE HAVE FUNCTOR

$\mathcal{C}^{op} \times \mathcal{C} \xrightarrow{\mathcal{C}(-,-)} \text{Set}$

$(X, Y) \longmapsto \mathcal{C}(X, Y)$

THIS IS "CONTRAVARIANT" IN X
AND COVARIANT IN Y .

TERMINOLOGY

A FUNCTOR $\mathcal{C} \xrightarrow{F} \mathcal{D}$ IS

FAITHFUL THE INDUCED

MAP $\mathcal{C}(X, Y) \longrightarrow \mathcal{D}(FX, FY)$

IS 1-1 FOR ALL X AND Y

IT IS FULLY FAITHFUL

IF THIS MAP IS ALWAYS
A BIJECTION.

A SUBCATEGORY \mathcal{C}' OF \mathcal{C}

IS A CATEGORY SUCH THAT

1) EACH OBJECT OF \mathcal{C}' IS
AND OBJECT OF \mathcal{C}

2) SAME FOR MORPHISMS

EXAMPLE

a) $\text{FSet} = \text{CATEGORY OF FINITE SETS}$

IT IS A ^{FULL} SUBCATEGORY OF
 Set .

b) $\mathcal{C} = \text{ANY CATEGORY}$

\mathcal{C}' IS THE ^{WIDE} SUBCATEGORY,
HAVING THE SAME OBJECTS
AS \mathcal{C} , BUT THE ONLY
MORPHISMS ARE ISOS.

\mathcal{C}'' IS THE ^{WIDE} SUBCATEGORY
WITH THE SAME OBJECTS
BUT ONLY IDENTITY
MORPHISMS

A CATEGORY IS DISCRETE
IF ALL MORPHISMS ARE
IDENTITIES.

DEF A SUBCATEGORY $\mathcal{C} \subset \mathcal{D}$
IS FULL IF FOR EACH
PAIR OF OBJECTS $X, Y \in \mathcal{C}$,
 $\mathcal{C}(X, Y) = \mathcal{D}(X, Y)$. IT IS
WIDE (OR LLUF) IF
IT CONTAINS ALL OBJECTS
OF \mathcal{D} .

AMENITIES THAT A CATEGORY
MAY OR MAY NOT HAVE.

1) AN INITIAL OBJECT X IN \mathcal{C}
HAS THE PROPERTY THAT
 $\mathcal{C}(X, Y)$ IS A SINGLETON

FOR ALL OBJECTS Y
2) A TERMINAL OBJECT X IN \mathcal{C}
HAS $\mathcal{C}(W, X) = \text{SINGLETON}$
FOR ALL $W \in \mathcal{C}$.

EXERCISE: ANY TWO INITIAL
(TERMINAL) ARE ISOMORPHIC.

EXAMPLE $A \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} B$ HAS NEITHER

DEF A CATEGORY IS POINTED
IF ITS INITIAL AND TERMINAL
OBJECTS ARE THE SAME.

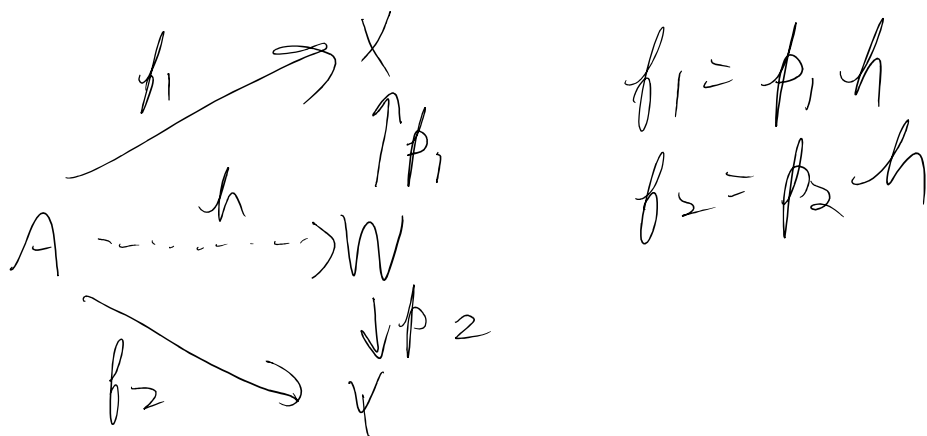
EG. CATEGORIES OF
POINTED SPACES OR SETS.

3) PRODUCTS FOR OBJECTS
 X AND Y IN \mathcal{C} , THE PRODUCT
 $W = "X \times Y"$ HAS THE FOLLOWING
PROPERTIES

a) THE ARE MORPHISMS

$$\begin{array}{ccc}
 & \xrightarrow{p_1} & X \\
 W & \xrightarrow{p_2} & Y
 \end{array}$$

b) GIVEN MAPS $A \xrightarrow{p_1} X$
 AND $A \xrightarrow{p_2} Y$, $\exists!$ $A \xrightarrow{h} W$
 SUCH THAT

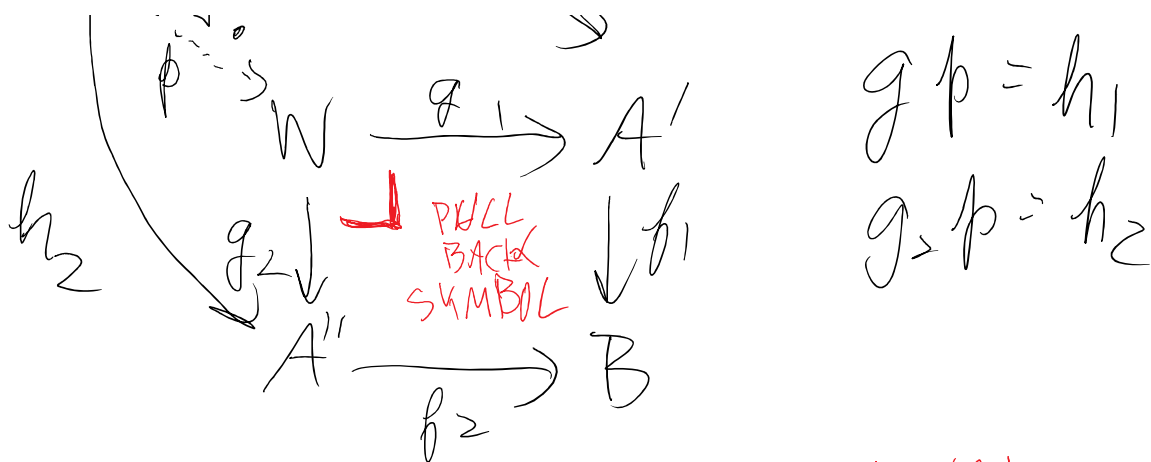


4) COPRODUCTS AS ABOVE
 WITH ARROWS REVERSED.
 (DISJOINT UNION IN SET)

5) PULLBACKS GIVEN

$$\begin{array}{ccc}
 X & \xrightarrow{h_1} & B \subset A'' \\
 \downarrow \exists! p & & \downarrow f_2 \\
 & &
 \end{array}$$

$q \circ p = h_1$



PULLBACK = PRODUCT WHEN $B = \text{TERMINAL OBJECT}$

WE WANT AN OBJECT W

WITH MAPS g_1 AND g_2

SUCH THAT $b_2 g_2 = b_1 g_1$ SUCH

THAT FOR ANY X WITH

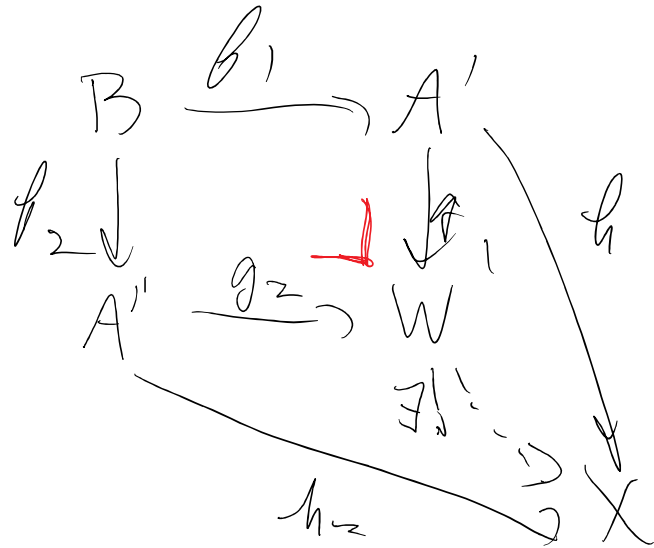
MAPS $X \xrightarrow{h_1} A'$, $X \xrightarrow{h_2} A''$

WITH $b_1 h_1 = b_2 h_2$

EXAMPLE IN SET

$$W = \{ (a', a'') \in A' \times A'' : f_1(a') = f_2(a'') \in B \}$$

6) PUSHOUTS AS ABOVE WITH
ARROWS REVERSED



IN SET

$$W = A' \amalg A'' / \left(\beta_1(h) \in A' \sim f_2(h \in A'') \right)$$

NEXT NATURAL
TRANSFORMATIONS!