MORE ABOUT FUNCTORS

RECALL A FUNCTOR

$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$

ASSIGNS TO EACH OBJECT X IN $\mathcal{C}$
AN OBJECT FX IN $\mathcal{D}$
AND TO EACH MORPHISM $X \xrightarrow{g} Y$
IN $\mathcal{C}$, A MORPHISM $Fg : FX \to FY$ in $\mathcal{D}$

$\mathcal{D}$

$$\begin{array}{c} FX \\ \downarrow{\scriptstyle Fg} \\ FY \end{array}$$

IT PRESERVES MORPHISM COMPOSITION AND IDENTITY MORPHISMS

EXAMPLES

1) LET G BE A GROUP
$\mathcal{B}G$ IS THE ONE OBJECT WITH

A MORPHISM FOR EACH $g \in G$.
A FUNCTOR $BG \xrightarrow{F} \mathcal{C}$
DETERMINES AN OBJECT X IN $\mathcal{C}$
AND A COLLECTION OF AUTOMORPHISMS OF X RELATED TO G. THIS DEFINES AN ACTION OF G ON X.

2) LET X BE AN OBJECT IN A CATEGORY $\mathcal{C}$. CONSIDER

$$\mathcal{C} \xrightarrow{\mathcal{C}(X,-)} \mathcal{S}et$$

$$Y \longmapsto \mathcal{C}(X,Y) =$$ SET OF MORPHISMS $X \to Y$ IN $\mathcal{C}$.

FUNCTOR COREPRESENTED BY X

$$\begin{array}{ccccc}
X \longrightarrow Y & \qquad & \mathcal{C}(X,Y) & \qquad & f : X \to Y \\
\big\downarrow g & & \big\downarrow g_* & & \big\downarrow \\
Z & & \mathcal{C}(X,Z) & & gf : X \to Z
\end{array}$$

$\tilde{Z}$ $\qquad$ $\mathcal{C}(X, Z)$ $\qquad$ $gf : X \to Z$

3) CONSIDER

$$\mathcal{C} \xrightarrow{\mathcal{C}(-,X)} \mathcal{S}et$$

$$W \longmapsto \mathcal{C}(W, X)$$

FUNCTOR REPRESENTED BY X

$$\begin{tikzcd}
W \arrow[d, "g"'] & \mathcal{C}(W, X) \ni fg \\
Z & \mathcal{C}(Z, X) \ni f \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
W \arrow[d, "g"'] \arrow[dr, "fg"] & \\
Z \arrow[r, "f"'] & X
\end{tikzcd}$$

THIS "FUNCTOR" IS CONTRAVARIANT,
I.E. ARROW REVERSING
IT IS A FUNCTOR

$$\mathcal{C}^{op} \xrightarrow{\mathcal{C}(-,X)} \mathcal{S}et$$

$$\begin{tikzcd}
W \\
Z \arrow[u, "g"]
\end{tikzcd}$$

THIS FUNCTOR PRESERVES

2 DIRECTION OF ARROWS.

WE CAN COMBINE THESE LAST TWO INTO A SINGLE FUNCTOR.

<u>DEFINITION</u> LET $\mathcal{C}$ AND $\mathcal{D}$ BE CATEGORIES. THEIR <u>PRODUCT CATEGORY</u> $\mathcal{C} \times \mathcal{D}$ HAS AS OBJECTS PAIRS (X, Y) WITH $X \in \mathcal{C}$ AND $Y \in \mathcal{D}$ AND MORPHISMS $(X, Y) \to (X', Y')$ ARE PAIRS (f, g) WHERE

$f: X \to X'$ IS A MORPHISM IN $\mathcal{C}$

$g: Y \to Y$ " " $\mathcal{D}$

COMPOSITION IS OBVIOUS.

4) WE HAVE FUNCTOR

$$\mathcal{C}^{op} \times \mathcal{C} \xrightarrow{\mathcal{C}(-,-)} \mathcal{S}et$$

$$(X, Y) \longmapsto \mathcal{C}(X, Y)$$

THIS IS "CONTRAVARIANT" IN X AND COVARIANT IN Y.

TERMINOLOGY

A FUNCTOR $\mathcal{C} \xrightarrow{F} \mathcal{D}$ IS *FAITHFUL* THE INDUCED MAP $\mathcal{C}(X,Y) \longrightarrow \mathcal{D}(FX, FY)$ IS 1-1 FOR ALL X AND Y

IT IS *FULLY FAITHFUL* IF THIS MAP IS ALWAYS A BIJECTION.

A *SUBCATEGORY* $\mathcal{C}'$ OF $\mathcal{C}$ IS A CATEGORY SUCH THAT

1) EACH OBJECT OF $\mathcal{C}'$ IS AND OBJECT OF $\mathcal{C}$

2) SAME FOR MORPHISMS

EXAMPLE

a) $FSet$ = CATEGORY OF FINITE SETS

IT IS A FULL SUBCATEGORY OF Set.

b) $\mathcal{C}$ = ANY CATEGORY

$\mathcal{C}'$ IS THE WIDE SUBCATEGORY, HAVING THE SAME OBJECTS AS $\mathcal{C}$, BUT THE ONLY MORPHISMS ARE ISOS.

$\mathcal{C}''$ IS THE WIDE SUBCATEGORY WITH THE SAME OBJECTS BUT ONLY IDENTITY MORPHISMS

A CATEGORY IS DISCRETE IF ALL MORPHISMS ARE IDENTITIES.

DEF A SUBCATEGORY $\mathcal{C} \subset \mathcal{D}$ IS FULL IF FOR EACH PAIR OF OBJECTS $X, Y \in \mathcal{C}$, $\mathcal{C}(X,Y) = \mathcal{D}(X,Y)$. IT IS WIDE (OR LLUF) IF IT CONTAINS ALL OBJECTS OF $\mathcal{D}$.

AMENITIES THAT A CATEGORY MAY OR MAY NOT HAVE.

1) AN INITIAL OBJECT X IN $\mathcal{C}$ HAS THE PROPERTY THAT $\mathcal{C}(X,Y)$ IS A SINGLETON

FOR ALL OBJECTS Y

2) A <u>TERMINAL OBJECT</u> X IN $\mathcal{C}$ HAS $\mathcal{C}(W, X) = $ SINGLETON FOR ALL $W \in \mathcal{C}$.

EXERCISE: ANY TWO INITIAL (TERMINAL) ARE ISOMORPHIC.

EXAMPLE $A \rightrightarrows B$ HAS NEITHER

DEF A CATEGORY IS <u>POINTED</u> IF ITS INITIAL AND TERMINAL OBJECTS ARE THE SAME.

EG. CATEGORIES OF POINTED SPACES OR SETS.

3) <u>PRODUCTS</u> FOR OBJECTS X AND Y IN $\mathcal{C}$, THE PRODUCT $W = $ "$X \times Y$" HAS THE FOLLOWING PROPERTIES

a) THE ARE MORPHISMS

$$\begin{tikzcd} & X \\ W \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & \\ & Y \end{tikzcd}$$

b) GIVEN MAPS $A \xrightarrow{f_1} X$ AND $A \xrightarrow{f_2} Y$, $\exists!\ A \xrightarrow{h} W$ SUCH THAT

$$\begin{tikzcd} & X \\ A \arrow[ur, "f_1"] \arrow[r, dashed, "h"] \arrow[dr, "f_2"'] & W \arrow[u, "p_1"'] \arrow[d, "p_2"] \\ & Y \end{tikzcd}$$

$f_1 = p_1 h$

$f_2 = p_2 h$

4) <u>COPRODUCTS</u> AS ABOVE WITH ARROWS REVERSED.

(DISJOINT UNION IN <u>SET</u>)

5) <u>PULLBACKS</u> GIVEN $A' \xrightarrow[f_1]{} B \xleftarrow[f_2]{} A''$

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "h_1"] & & \\ & \cdot & \cdot \end{tikzcd}$$

p

$q p = h_1$

$$
\begin{tikzcd}
W \arrow[r, "g_1"] \arrow[d, "g_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[d, "f_1"] \\
A'' \arrow[r, "f_2"'] & B
\end{tikzcd}
$$

PULL BACK SYMBOL

p, h_2

$g_1 p = h_1$

$g_2 p = h_2$

PULLBACK = PRODUCT WHEN B = TERMINAL OBJECT

WE WANT AN OBJECT W WITH MAPS g_1 AND g_2 SUCH THAT $f_2 g_2 = f_1 g_1$ SUCH THAT FOR ANY X WITH MAPS $X \xrightarrow{h_1} A'$, $X \xrightarrow{h_2} A''$ WITH $f_1 h_1 = f_2 h_2$

EXAMPLE IN $\underline{\text{SET}}$

$$W = \{(a', a'') \in A' \times A'' : f_1(a') = f_2(a'') \in B\}$$

6) $\underline{\text{PUSHOUTS}}$ AS ABOVE WITH ARROWS REVERSED

$$
\begin{tikzcd}
B \arrow[r, "f_1"] \arrow[d, "f_2"'] & A' \arrow[d, "g_1"] \arrow[ddr, bend left, "h_1"] & \\
A'' \arrow[r, "g_2"] \arrow[drr, bend right, "h_2"'] \arrow[ur, phantom, "\lrcorner"] & W \arrow[dr, dashed, "\exists!"] & \\
 & & X
\end{tikzcd}
$$

IN $\underline{\text{SET}}$

$$W = A' \amalg A'' \Big/ \big(f_1(b) \in A' \sim f_2(b) \in A'' \big)$$

$\underline{\text{NEXT}}$ NATURAL TRANSFORMATIONS!