NATURAL TRANSFORMATIONS

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SUPPOSE WE HAVE FUNCTORS

$$X \qquad \begin{tikzcd}[column sep=huge] \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{D} \end{tikzcd} \qquad \begin{tikzcd} FX \arrow[d, "\theta_X"] \\ GX \end{tikzcd}$$

DEF: A NATURAL TRANSFORMATION $\theta : F \Rightarrow G$ CONSISTS OF A MORPHISM $\theta_X : FX \longrightarrow GX$ IN $\mathcal{D}$ FOR EACH OBJECT IN $\mathcal{C}$ SUCH THAT FOR EACH MORPHISM $X \xrightarrow{f} Y$ IN $\mathcal{C}$, THE DIAGRAM

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "\theta_X"'] & FY \arrow[d, "\theta_Y"] \\ GX \arrow[r, "Gf"] & GY \end{tikzcd}$$

COMMUTES IN $\mathcal{D}$.

REMARK: IN CATEGORY, DEFINITIONS RARELY REQUIRE TWO OBJECTS TO BE EQUAL, BUT RATHER TO BE ISOMORPHIC. DEFINITIONS REQUIRING EQUALITY ARE SAID TO BE EVIL. OTH, WE

OFTEN REQUIRE MORPHISMS TO BE EQUAL, I.E. THE TWO ELEMENTS IN THE MORPHISM SET ARE THE SAME.

RELATED <u>DEF</u>: θ AS ABOVE IS A <u>NATURAL EQUIVALENCE</u> IF EACH MORPHISM $\theta_X : FX \to GX$ IS AN ISOMORPHISM IN $\mathcal{D}$.

<u>DEF</u> TWO CATEGORIES $\mathcal{C}$ AND $\mathcal{D}$ ARE <u>EQUIVALENT</u> IF THERE ARE FUNCTORS

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd}$$

SUCH THAT THERE ARE NATURAL EQUIVALENCES

$$GF \Longrightarrow 1_{\mathcal{C}}$$

AND $FG \Longrightarrow 1_{\mathcal{D}}$.

IF GF AND FG ARE THE IDENTITY FUNCTORS, WE $\mathcal{C}$ AND $\mathcal{D}$ ARE ISOMORPHIC CATEGORIES.

<u>EXAMPLE</u> k = FIELD

$Vect_k$ = CATEGORY OF k-VECTOR SPACES

$Vect_k^{fin}$ = CATEGORY OF FINITE DIMENSIONAL k-VECTOR SPACES

$\mathcal{C}$ = CATEGORY WHOSE OBJECTS ARE NATURAL #s n WITH $\mathcal{C}(m,n) = Vect(k^m, k^n)$

THERE ARE FUNCTORS

$$\mathcal{C} \xrightarrow{F} \underset{\mathcal{D}}{Vect_k^{fin}} \xrightarrow{G?} \mathcal{C}$$

$$n \longmapsto k^n$$

$$V \longmapsto \dim(V)$$

$GF = 1_{\mathcal{C}}$, BUT $FG \neq I_{\mathcal{D}}$

$\mathcal{C}$ IS A <u>SKELETON</u> OF $\mathcal{D} = Vect_k^{fin}$, I.E. EACH OBJECT IS ISOMORPHIC TO A UNIQUE OBJECT IN $\mathcal{C}$.

EXAMPLES OF NATURAL TRANSFORMATIONS FROM ER 1,4,3

i) DOUBLE DUAL FUNCTOR

$$Vect_k \longrightarrow Vect_k$$

$$V \longmapsto (V^*)^* = V^{**}$$

THERE IS A MAP $ev : V \longmapsto V^{**}$

$$v \longmapsto ev : V^* \to k$$

WE KNOW FOR ANY MORPHISM
$\phi : V \longrightarrow W$ IN $\mathrm{Vect}_{\mathbb{R}}$

$$\begin{tikzcd}
V \arrow[r, "ev"] \arrow[d, "\phi"'] & V^{**} \arrow[d, "\phi^{**}"] \\
W \arrow[r, "ev"] & W^{**}
\end{tikzcd}$$

HENCE $ev : 1_{\mathrm{Vect}_{\mathbb{R}}} \Longrightarrow (\ )^{**}$

iii) LET $P : \mathcal{Set} \longrightarrow \mathcal{Set}$

$S \longmapsto P(S)$

$P(S)$ = SET OF ALL SUBSETS OF S = POWER SET OF S

THERE IS A NATURAL TRANSFORMATION

$$1_{SET} \xRightarrow{\ \eta\ } P$$

$$A = 1_{SET}(A) \xrightarrow{\eta_A} P(A)$$

$$A \ni a \longmapsto \{a\} \subset A$$

iv) FOR A GROUP G, BG IS THE ONE OBJECT WITH A ENDOMORPHISM FOR EACH $\gamma \in G$ WITH COMPOSITION CORRESPONDS TO GROUP MULTIPLICATION.

A FUNCTOR $BG \xrightarrow{X} \mathcal{D}$ DEFINES AN ACTION OF G ON SOME OBJ

A NATURAL TRANSFORMATION $\theta : X \Rightarrow Y$ BETWEEN SUCH FUNCTORS IS A MORPHISM $X \xrightarrow{\theta} Y$ (OBJECTS IN $\mathcal{D}$) WHICH RESPECTS THE G-ACTION

IE

$$\begin{tikzcd} X \arrow[r, "\theta"] \arrow[d, "\gamma"'] & Y \arrow[d, "\gamma"] \\ {} \arrow[r, "\theta"] & {} \end{tikzcd}$$

COMMUTES $\forall \gamma \in G$

IE $\gamma \downarrow \qquad \downarrow \gamma$

$X \xrightarrow{\theta} Y$ COMMUTES $\forall\, \gamma \in G$.

FUNCTORIAL FACTORIZATIONS

<u>EXAMPLE.</u> IN SET ANY MORPHISM

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\text{EPI}"'] & & Y \\ & f(X) \arrow[ur, "\text{MONO}"'] & \end{tikzcd}$$

THIS IS A FACTORIZATION OF f.

SOME LITTLE CATEGORIES

$[0] = \{0\}$

$[1] = \{0 \longrightarrow 1\}$

$[2] = \{0 \longrightarrow 1 \longrightarrow 2\}$

THESE ARE CATEGORIES AND ORDERED SETS

A FUNCTOR $[1] \longrightarrow \mathcal{C}$ DEFINES A MORPHISM $X_0 \longrightarrow X_1$

A FUNCTOR $[2] \longrightarrow \mathcal{C}$ INC

DEFINES A DIAG $X_0 \longrightarrow X_1 \longrightarrow X_2$

IN $\mathcal{C}$.

FOR $0 \le j \le 2$, LET $d_j : [1] \to [2]$ DENOTE THE ORDER PRESERVING MONOMORPHISM NOT HAVING j IN ITS IMAGE. IT IS ALSO A FUNCTOR.

<u>NOTATION</u> FOR A SMALL CATEGORY J (e.g. $[n]$) $\mathcal{C}^J$ DENOTES THE CATEGORY OF FUNCTORS $J \longrightarrow \mathcal{C}$

ITS OBJECTS ARE FUNCTOR $J \to \mathcal{C}$

" MORPHISMS ARE NATURAL TRANSFORMATIONS BETWEEN THEM

$$[1] \xrightarrow{d_j} [2] \qquad 0 \le j \le 2$$

INDUCES $\mathcal{C}^{[1]} \xleftarrow{\delta_j} \mathcal{C}^{[2]}$

DEF A FUNCTORIAL FACTORIZATION IN $\mathcal{C}$ IS A FUNCTOR

$$\begin{tikzcd} \mathcal{C}^{[1]} \arrow[r, "F"] & \mathcal{C}^{[2]} \arrow[r, shift left, "\delta_2"] \arrow[r, shift right, "\delta_0"'] & \mathcal{C}^{[1]} \end{tikzcd}$$

$$X_0 \xrightarrow{f} X_1 \qquad \begin{tikzcd} X_0 \arrow[r, "\delta_2 F(f)"] \arrow[rr, bend right, "f"'] & X_{1/2} \arrow[r, "\delta_0 F(f)"] & X_1 \end{tikzcd}$$

THIS IS IMPORTANT IN MODEL CATEGORES.

NEXT TOPIC: YONEDA LEMMA.