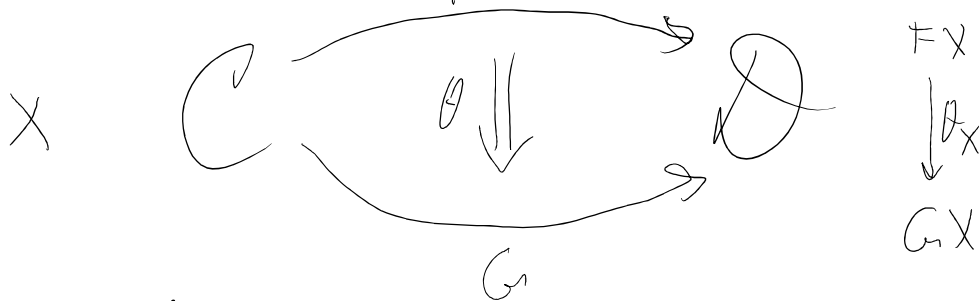


# NATURAL TRANSFORMATIONS

Monday, February 8, 2021 1:58 PM

SUPPOSE WE HAVE FUNCTORS



DEF: A NATURAL TRANSFORMATION

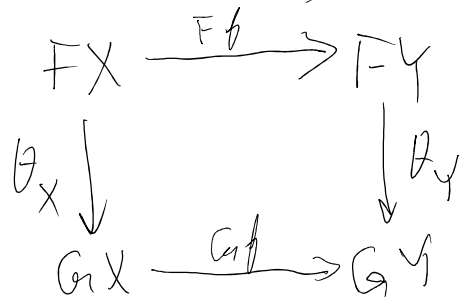
$\theta: F \Rightarrow G$  CONSISTS OF

A MORPHISM  $\theta_x: FX \rightarrow GX$  IN  $\mathcal{D}$

FOR EACH OBJECT IN  $\mathcal{C}$

SUCH THAT FOR EACH MORPHISM

$X \xrightarrow{\phi} Y$  IN  $\mathcal{C}$ , THE DIAGRAM



COMMUTES IN  $\mathcal{D}$ .

REMARK: IN CATEGORY DEFINITIONS

RARELY REQUIRE TWO OBJECTS TO BE EQUAL, BUT RATHER TO BE

ISOMORPHIC. DEFINITIONS

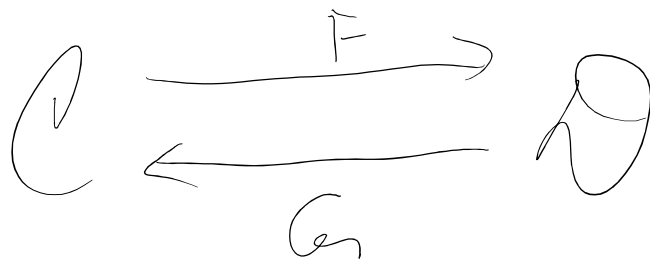
REQUIRING EQUALITY ARE SAID

TO BE EVIL. OTH, WE

OFTEN REQUIRE MORPHISMS TO BE EQUAL, I.E. THE TWO ELEMENTS IN THE MORPHISM SET ARE THE SAME.

RELATED DEF:  $\mathcal{C}$  AS ABOVE IS A NATURAL EQUIVALENCE IF EACH MORPHISM  $\theta_x: FX \rightarrow GX$  IS AN ISOMORPHISM. IN  $\mathcal{D}$ .

DEF TWO CATEGORIES  $\mathcal{C}$  AND  $\mathcal{D}$  ARE EQUIVALENT IF THERE ARE FUNCTORS



SUCH THAT THERE ARE NATURAL EQUIVALENCES

$$GF \implies 1_{\mathcal{C}}$$

$$\text{AND } FG \implies 1_{\mathcal{D}}$$

IF  $G_F$  AND  $F_G$  ARE THE IDENTITY FUNCTORS, WE  $\mathcal{C}$  AND  $\mathcal{D}$  ARE ISOMORPHIC CATEGORIES.

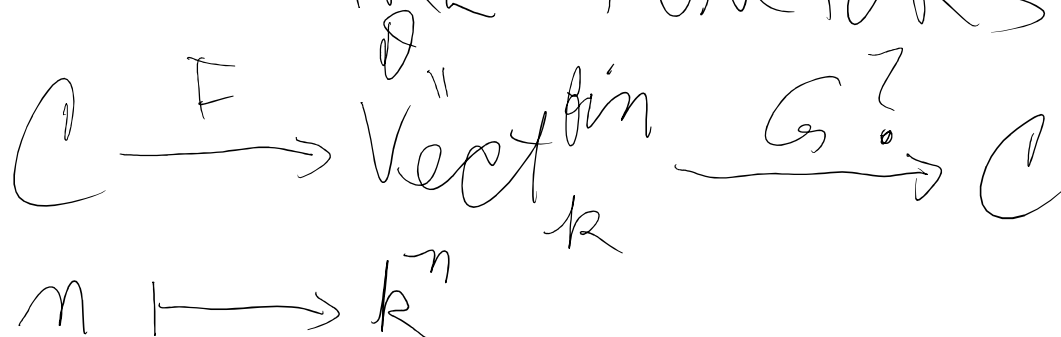
EXAMPLE  $k = \text{FIELD}$

$\text{Vect}_k = \text{CATEGORY OF } k\text{-VECTOR SPACES}$

$\text{Vect}_k^{\text{fin}} = \text{CATEGORY OF FINITE DIMENSIONAL } k\text{-VECTOR SPACES}$

$\mathcal{C} = \text{CATEGORY WHOSE OBJECTS ARE NATURAL \#s } n$   
 WITH  $\mathcal{C}(m, n) = \text{Vect}(k^m, k^n)$

THERE ARE FUNCTORS



$$V \longmapsto \dim(V)$$

$$GF = I_{\mathcal{C}}, \text{ BUT } FG \neq I_{\mathcal{D}}$$

$\mathcal{C}$  IS A SKELETON OF

$\mathcal{D} = \text{Vect}_{\mathbb{R}}^{\text{fin}}$ , I.E. EACH OBJECT IS ISOMORPHIC TO A UNIQUE OBJECT IN  $\mathcal{C}$ .

EXAMPLES OF NATURAL TRANSFORMATIONS FROM  $\text{Fin} \rightarrow \text{Fin}$

i) DOUBLE DUAL FUNCTOR

$$\text{Vect}_{\mathbb{R}} \longrightarrow \text{Vect}_{\mathbb{R}}$$

$$V \longmapsto (V^*)^* = V^{**}$$

THERE IS A NAT  $ev: V \rightarrow V^{**}$

$$v \mapsto ev: V \rightarrow V^{**}$$

WE KNOW FOR ANY MORPHISM  $\tilde{v}$

$$\phi: V \rightarrow W \text{ IN } \text{Vect}_R$$

$$\begin{array}{ccc} V & \xrightarrow{\text{ev}} & V^{**} \\ \phi \downarrow & & \downarrow \phi^{**} \\ W & \xrightarrow{\text{ev}} & W^{**} \end{array}$$

HENCE  $\text{ev}: \text{Vect}_R \Rightarrow (\ )^{**}$

(ii) LET  $P: \text{Set} \rightarrow \text{Set}$

$$S \mapsto P(S)$$

SET OF ALL SUBSETS OF  $S$  = POWER SET OF  $S$

THERE IS A NATURAL TRANSFORMATION

$$I_{\text{SET}} \xrightarrow{\eta} P$$

$$A = I_{\text{SET}}(A) \xrightarrow{\eta_A} P(A)$$

$$A \ni a \longmapsto \{a\} \subset A$$

iv) FOR A GROUP  $G$ ,  $BG$  IS THE ONE OBJECT WITH A ENDOMORPHISM FOR EACH  $\gamma \in G$  WITH COMPOSITION CORRESPONDS TO GROUP MULTIPLICATION.

A FUNCTOR  $BG \xrightarrow{X} \mathcal{D}$  DEFINES AN ACTION OF  $G$  ON SOME OBJ

A NATURAL TRANSFORMATION

$$\theta : X \Rightarrow Y \quad \text{BETWEEN}$$

SUCH FUNCTORS IS

$$\text{A MORPHISM } X \xrightarrow{\theta} Y \quad (\text{OBJECTS IN } \mathcal{D})$$

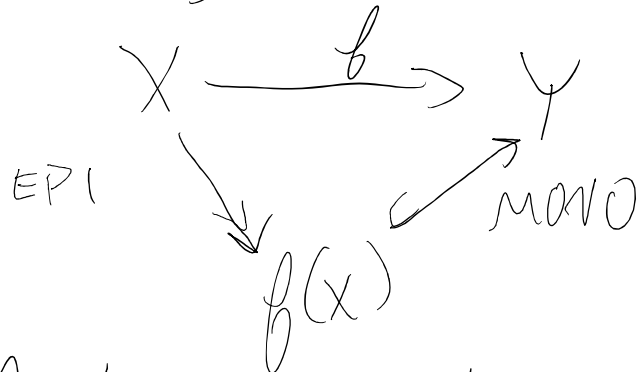
WHICH RESPECTS THE  $G$ -ACTION

$$\begin{array}{ccc} X & \xrightarrow{\theta} & Y \\ \gamma \downarrow & & \downarrow \gamma \\ X & \xrightarrow{\theta} & Y \end{array} \quad \text{COMMUTES}$$

$$\begin{array}{ccc}
 \gamma \downarrow & & \downarrow \gamma \\
 X & \xrightarrow{\alpha} & Y
 \end{array}
 \quad \text{COMMUTES}$$

$\forall \gamma \in G.$

FUNCTORIAL FACTORIZATIONS  
 EXAMPLE. IN SET ANY MORPHISM



THIS IS A FACTORIZATION OF  $f$ .

SOME LITTLE CATEGORIES

$$[0] = \{0\}$$

$$[1] = \{0 \rightarrow 1\}$$

$$[2] = \{0 \rightarrow 1 \rightarrow 2\}$$

THESE ARE CATEGORIES AND ORDERED SETS

A FUNCTOR  $[1] \rightarrow \mathcal{C}$

DEFINES A MORPHISM  $X_0 \rightarrow X_1$

A FUNCTOR  $[2] \rightarrow \mathcal{C}$

INC

DEFINES A DIAG

$$X_0 \rightarrow X_1 \rightarrow X_2$$

In  $\mathcal{C}$ .

FOR  $0 \leq j \leq 2$ , LET  $d_j: [1] \rightarrow [2]$   
DENOTE THE ORDER PRESERVING  
MONOMORPHISM NOT HAVING  $j$   
IN ITS IMAGE. IT IS ALSO  
A FUNCTOR.

NOTATION FOR A SMALL CATEGORY  
 $\mathcal{J}$  (e.g.  $[n]$ )  $\mathcal{C}^{\mathcal{J}}$  DENOTES  
THE CATEGORY OF FUNCTORS  
 $\mathcal{J} \rightarrow \mathcal{C}$

ITS OBJECTS ARE FUNCTOR  $\mathcal{J} \rightarrow \mathcal{C}$   
" MORPHISMS ARE  
NATURAL TRANSFORMATIONS  
BETWEEN THEM

$$[1] \xrightarrow{d_j} [2] \quad 0 \leq j \leq 2$$

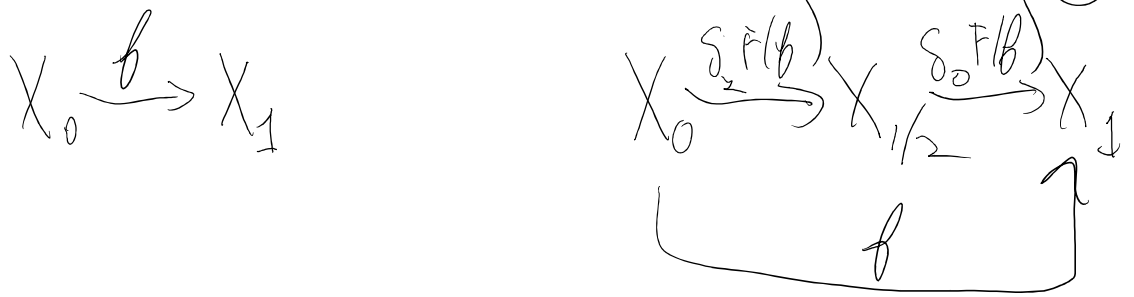
INDUCES

$$\mathcal{C}^{[1]} \xleftarrow{d_j} \mathcal{C}^{[2]}$$



DEF A FUNCTORIAL FACTORIZATION

IN  $\mathcal{C}$  IS A FUNCTOR  $\mathcal{C}[\Gamma] \xrightarrow{F} \mathcal{C}[\Sigma] \xrightarrow{S_0} \mathcal{C}[\Gamma]$



THIS IS IMPORTANT  
IN MODEL CATEGORIES.

NEXT TOPIC: YONEDA LEMMA.