SYMMETRIC MONOIDAL CATEGORIES

Monday, March 15, 2021 1:56 PM

RECALL A GROUP IS A SET G WITH A BINARY OPERATION $G \times G \to G$ WHICH IS

$e, \gamma \in G$

1) ASSOCIATIVE
2) UNITAL (WITH IDENTITY ELEMENT e) $e\gamma = \gamma = \gamma e$
3) EVERY ELEMENT HAS AN INVERSE

A MONOID IS A SET WITH BINARY OPERATION SATISFYING 1) AND 2).

IF THE OPERATION IS COMMUTATIVE, THE GROUP OR MONOID IS ABELIAN

E.G. $(\mathbb{N}, +, 0)$ (NATURAL #S UNDER ADDITION) IS AN ABELIAN MONOID.

DEF A MONOIDAL STRUCTURE $\otimes$ ON A CATEGORY $\mathcal{C}$ IS A FUNCTOR

$$\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$$

$$(X, Y) \longmapsto X \otimes Y$$

AND A UNIT OBJECT $\mathbb{1}$ ALONG WITH

1) ASSOCIATIVITY: FOR OBJECTS X, Y, Z, THERE IS A NATURAL ISOMORPHISM

ASSOCIATOR

$$C \quad : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$$

ASSOCIATOR

$$a_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$$

IT IS PART OF A NATURAL EQUIVALENCE BETWEEN TWO FUNCTORS

$$\begin{tikzcd}
\mathcal{C} \times \mathcal{C} \times \mathcal{C} \arrow[r, "1_{\mathcal{C}} \times \otimes"] \arrow[d, "\otimes \times 1_{\mathcal{C}}"'] & \mathcal{C} \times \mathcal{C} \arrow[d, "\otimes"] \arrow[dl, Rightarrow, "a"'] \\
\mathcal{C} \times \mathcal{C} \arrow[r, "\otimes"'] & \mathcal{C}
\end{tikzcd}$$

$$\begin{tikzcd}
(X, Y, Z) \arrow[r, mapsto] \arrow[d, mapsto] & (X, Y \otimes Z) \arrow[d, mapsto] \\
(X \otimes Y, Z) \arrow[r, mapsto] & (X \otimes Y) \otimes Z \qquad X \otimes (Y \otimes Z)
\end{tikzcd}$$

2) UNITAL CONDITION: FOR ANY OBJECT X THERE ARE NATURAL ISOMORPHISMS

$$1 \otimes X \xrightarrow[\lambda_X]{\simeq} X \xleftarrow[\rho_X]{\simeq} X \otimes 1$$

LEFT UNITOR — RIGHT UNITOR

THESE ARE REQUIRED TO SATISFY

a) THE ISOMORPHISMS λ_1 AND ρ_1 ARE THE SAME.

ARE THE SAME.

b) IN 1) WHEN $Y = 1$, THE FOLLOWING COMMUTES

$$\begin{tikzcd}
X \otimes (1 \otimes Y) \arrow[rr, "a_{X,1,Y}", "\cong"'] \arrow[dr, "X \otimes \lambda_Y"', "\cong"] & & (X \otimes 1) \otimes Y \arrow[dl, "\rho_X \otimes Y", "\cong"'] \\
& X \otimes Y &
\end{tikzcd}$$

c) STASHEFF PENTAGON FOR $(W, X, Y, Z) \in \mathcal{C} \times \mathcal{C} \times \mathcal{C} \times \mathcal{C}$

$$\begin{tikzcd}
& (W \otimes X) \otimes (Y \otimes Z) \arrow[dr, "a_{W \otimes X, Y, Z}"] & \\
W \otimes (X \otimes (Y \otimes Z)) \arrow[ur, "a_{W,X,Y \otimes Z}"] \arrow[d, "W \otimes a_{X,Y,Z}"'] & & ((W \otimes X) \otimes Y) \otimes Z \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[r, "a_{W, X \otimes Y, Z}"'] & (W \otimes (X \otimes Y)) \otimes Z \arrow[ur, "a_{W,X,Y} \otimes Z"'] &
\end{tikzcd}$$

THIS MUST COMMUTE

REMARK IF $\mathcal{C}$ IS COMPLETE, IT HAS A PRODUCT AND A

IT HAS A PRODUCT AND A TERMINAL OBJECT WHICH IS THE UNIT FOR THE PRODUCT.
THIS IS THE CATEGORICAL OR CARTESIAN MONOIDAL STRUCTURE
DUALLY FOR COCOMPLETE CATEGORIES.

DEF A MONOIDAL STRUCTURE AS ABOVE IS <u>SYMMETRIC</u> IF THERE IS A NATURAL ISO

$$X \otimes Y \xrightarrow{\gamma_{X,Y}} Y \otimes X$$

SATISFYING

i) TRIANGLE IDENTITY

$$\begin{tikzcd}
1 \otimes X \arrow[rr, "\gamma_{1,X}"] \arrow[dr, "\lambda_X"'] & & X \otimes 1 \arrow[dl, "\rho_X"] \\
& {} &
\end{tikzcd}$$

COMMUTES

X

ii) FIRST HEXAGON IDENTITY

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[d, "\tau_{X,Y} \otimes Z"'] & X \otimes (Y \otimes Z) \arrow[l, "a_{X,Y,Z}"'] \arrow[r, "\tau_{X, Y \otimes Z}"] & (Y \otimes Z) \otimes X \\
(Y \otimes X) \otimes Z & Y \otimes (X \otimes Z) \arrow[l, "a_{Y,X,Z}"'] \arrow[r, "Y \otimes \tau_{X,Z}"] & Y \otimes (Z \otimes X) \arrow[u, "a_{Y,Z,X}"']
\end{tikzcd}$$

COMMUTES

PROP 2.6.11 (HHR) COEND REDUCTION

LET $(\mathcal{D}, \oplus, 0)$ BE A SMALL COCOMPLETE MONOIDAL CATEGORY. THEN FOR EACH $X, Y \in \mathcal{D}$

$$\int_{W \in \mathcal{D}} \mathcal{D}(W \oplus X, Y) \times \mathcal{D}(0, W) \cong \mathcal{D}(X, Y)$$

NOTE $\mathcal{D}(- \oplus X, Y) \times \mathcal{D}(0, -)$

IS A FUNCTOR $\mathcal{D}^{op} \times \mathcal{D} \longrightarrow \mathcal{S}et$

THERE IS A DUAL STATEMENT ABOUT ENDS.

DEF 2.6.19 LET $(\mathcal{C}, \oplus, 0)$ AND $(\mathcal{D}, \otimes, 1)$ BE (SYMMETRIC) MONOIDAL CATEGORIES. A FUNCTOR

$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$

NAIVE DEFINITION

"EVIL"

$$F(X \oplus Y) = F(X) \otimes F(Y)$$

F IS STRICTLY MONOIDAL

$\mathcal{D}$ IS THEN A $\mathcal{C}$-ALGEBRA (NOT A C^*-ALGEBRA)

F IS LAX (SYMM) MONOIDAL

IF THERE IS A NAT. TRANS

IF THERE IS A NAT. TRANS

$\mu : F(-) \otimes F(-) \Longrightarrow F(- \oplus -)$

AND A MORPHISM $F(0) \longrightarrow I$

IN $\mathcal{D}$, SATISFYING —

(THREE DIAGRAMS)

F IS STRONG MONOIDAL

IF μ IS A NAT. EQUIVALENCE

LAX IS EASIEST TO VERIFY AND IS WORTH MORE THAN YOU THINK.