Wednesday, March 24, 2021 1:50 PM

ENRICHED ENDS + COENDS

RECALL HOW COENDS ARE DEFINED

J = SMALL CATEGORY

$\mathcal{C}$ = COCOMPLETE "

$$H : J^{op} \times J \longrightarrow \mathcal{C} \rightsquigarrow \int_J H(j,j) \in \mathcal{C}$$

FOR EACH MORPHISM $x \xrightarrow{f} y$ IN J WE HAVE A DIAGRAM IN $\mathcal{C}$

NOTE REVERSAL

$$\begin{tikzcd}
H(y,x) \arrow[r, "f_*"] \arrow[d, "f^*"'] & H(y,y) \\
H(x,x) &
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
\coprod\limits_{\substack{f:x\to y \\ \text{in } J}} H(y,x) \arrow[r, "\varphi_*"] \arrow[d, "\varphi^*"] & \coprod\limits_{y \in J} H(y,y) \\
\coprod\limits_{x \in J} H(x,x) &
\end{tikzcd}$$

INDEXED BY SET OF MORPHISMS

INDEXED BY SET OF OBJECTS IN J

DEF 2.4.5 THE COEND $\int_J H(x,x)$ IS THE COEQUALIZER OF φ_* AND φ^*

IN THE ENRICHED SETTING, J AND $\mathcal{C}$ ARE $\mathcal{V}$-CATEGORIES, FOR A CLOSED SMC $(\mathcal{V}, \otimes, 1)$, SO $J(x,y)$ AND $\mathcal{C}(U,V)$

SO $J(x,y)$ AND $\mathcal{C}(U,V)$ ARE OBJECTS IN $\mathcal{V}$ INSTEAD OF SETS.

NOTE

DEPENDS ONLY ON x,y AND NOT ON $f : x \to y$

$$\coprod_{f : x \to y} H(y,x) = \coprod_{x,y \in J} \coprod_{J(x,y)} H(y,x)$$

$$= \coprod_{x,y} \underbrace{J(x,y)}_{\text{SET}} \times \underbrace{H(y,x)}_{\text{OBJECT IN } \mathcal{C}}$$

SINCE $\mathcal{C}$ IS COCOMPLETE, THE PRODUCT OF A SET AND AN OBJECT IN $\mathcal{C}$ IS DEFINED, I.E. $\mathcal{C}$ IS TENSORED OVER $\mathcal{S}et$.

IN THE ENRICHED SETTING, WE ASSUME $\mathcal{C}$ IS TENSORED OVER $\mathcal{V}$, IE WE CAN MAKE SENSE OF $V \otimes W$ FOR $V \in \mathcal{V}$, $W \in \mathcal{C}$

THUS WE HAVE

$$\coprod_{x,y \in J} \underline{J(x,y)} \otimes H(y,x)$$

OBJECT IN $\mathcal{V}$

OBJECT IN $\mathcal{C}$

WE STILL HAVE TWO MORPHISMS φ_* AND φ^* FROM THE ABOVE TO $\coprod_{x \in J} H(x,x)$.

THE ENRICHED COEND $\int_J H(x,x)$ IS THEIR COEQUALIZER.

EXAMPLE OF A SMC THAT IS <u>NOT</u> CLOSED

k = FIELD

$Vect_k$ = CATEGORY WHOSE OBJECTS ARE k-VECTOR SPACES

IF MORPHISMS ARE ALL LINEAR MAPS, WE GET A CLOSED SMC UNDER $\oplus$

SUPPOSE INSTEAD THAT OUR

SUPPOSE INSTEAD THAT OUR MORPHISMS ARE LINEAR EMBEDDINGS. THIS IS SYMMETRIC MONOIDAL UNDER $\oplus$

$\mathcal{E}mb_k$ IS <u>NOT CLOSED</u>.

BECAUSE THERE $\mathcal{E}mb_k(V,W)$ IS NOT A VECTOR SPACE.

THE DAY CONVOLUTION

$(\mathcal{V},\otimes,1)$ = CLOSED SMC, BICOMPLETE

$(\mathcal{D},\oplus,0)$ = SMALL SMC (POSSIBLY) CLOSED ENRICHED OVER $\mathcal{V}$

$[\mathcal{D},\mathcal{V}]$ = CATEGORY OF $\mathcal{V}$-FUNCTORS $\mathcal{D} \longrightarrow \mathcal{V}$

WE WILL SEE THAT THIS IS ALSO A BICOMPLETE CLOSED SMC.

THEOREM PROVED IN 1970 BY BRIAN DAY.

BRIAN DAY.

SUPPOSE $X, Y : \mathcal{D} \longrightarrow \mathcal{V}$

DEF 3.3.2

$$
\begin{tikzcd}
\mathcal{D} \times \mathcal{D} \arrow[r, "X \times Y"] \arrow[d, "\oplus"'] & \mathcal{V} \times \mathcal{V} \arrow[r, "\otimes"] & \mathcal{V} \\
\mathcal{D} \arrow[urr, dashed, "\text{LEFT KAN EXTENSION}"] & &
\end{tikzcd}
$$

$$\mathrm{Lan}_{\oplus}\left(\otimes(X \times Y)\right)$$

$$X \boxtimes Y$$

THIS IS THE ENRICHED COEND AS FOLLOWS

WILL DENOTED THE VALUE (IN $\mathcal{V}$) OF X OR Y ON $D \in \mathcal{D}$ BY X_D OR Y_D

THEN

OBJECTS IN $\mathcal{D}$

$$(X \boxtimes Y)_D = \int^{(A,B) \in \mathcal{D} \times \mathcal{D}} \mathcal{D}(A \oplus B, D) \otimes X_A \otimes Y_B$$

OBJECTS IN $\mathcal{V}$

NOTE THE FUNCTOR IS

$$(\mathcal{D} \times \mathcal{D})^{op} \times (\mathcal{D} \times \mathcal{D}) \xrightarrow{H} \mathcal{V}$$

$$\big((A,B),\ (A',B')\big) \longmapsto \mathcal{D}(A\oplus B, D)\otimes X_{A'}\otimes Y_{B'}$$

IN $\mathcal{V}$

CONTRAVARIANT IN (A,B)

COVARIANT IN (A',B')

THEOREM 3.3.5 THE BINARY OPERATION $\boxtimes : [\mathcal{D},\mathcal{V}]\times[\mathcal{D},\mathcal{V}]\to[\mathcal{D},\mathcal{V}]$ IS CLOSED SYMMETRIC MONOIDAL WITH UNIT

$$I = \text{⅄}^0 = \mathcal{D}(0,-)$$

$$I_D = \mathcal{D}(0,D) \quad \text{FOR } D\in\mathcal{D}$$

THE YONEDA FOR THE UNIT 0 OF $\mathcal{D}$.

CLOSED MEANS THE FUNCTOR

$$(-)\boxtimes X : [\mathcal{D},\mathcal{V}]\longrightarrow[\mathcal{D},\mathcal{V}]$$

HAS A RIGHT ADJOINT, THE INTERNAL HOM FUNCTOR

THE INTERNAL HOM FUNCTOR

IN $[\mathcal{D},\mathcal{V}]$ = CATEGORY OF FUNCTOR $\mathcal{D} \to \mathcal{V}$

IT IS DEFINED BY

$$\underline{[\mathcal{D},\mathcal{V}]}(X,Y)_D = \int^{(D_1,D_2)\in\mathcal{D}\times\mathcal{D}} \underbrace{\mathcal{V}(\underbrace{\mathcal{D}(D\oplus D_1, D_2)}_{\text{IN } \mathcal{D}}, \mathcal{V}(X_{D_1}, Y_{D_2}))}$$

$$= \int^{D_1\in\mathcal{D}} \underbrace{\mathcal{V}(X_{D_1}, Y_{D\oplus D_1})}_{\text{OBJECTS IN } \mathcal{V}}$$

$$X, Y \in [\mathcal{D},\mathcal{V}]$$

$$\underline{[\mathcal{D},\mathcal{V}]}(X,Y) \in [\mathcal{D},\mathcal{V}]$$

<u>EXAMPLE</u> 3.3.1

LET $A = \{ A_n : n \geq 0 \}$ GRADED SETS

$B = \{ B_n : n \geq 0 \}$ I.E. COLLECTIONS OF SETS

$\mathcal{B} = \{ B_n : n \geq 0 \}$ OF SETS INDEXED BY $\mathbb{N}$.

THEN THE GRADED SET $A \times B$ IS

$$(A \times B)_n = \coprod_{0 \leq i \leq n} (A_i \times B_{n-i})$$

REINTERPRETATION

LET $\mathcal{N}$ BE THE DISCRETE CATEGORY FOR $\mathbb{N}$.

A GRADED SET IS A FUNCTOR $\mathcal{N} \longrightarrow \mathcal{S}et$

NOTE $\mathcal{S}et$ IS A CLOSED BICOMPLETE SMC UNDER CARTESIAN PRODUCT

$\mathcal{N}$ IS SYMMETRIC MONOIDAL UNDER $+$ AND IS ENRICHED OF $\mathcal{S}et$

THEN FOR $A, B \in [\mathcal{N}, \mathcal{S}et]$

$$(A \times B)_n = \coprod_{i,j} A_i \times B_j \times \underline{\mathcal{N}(n, i+j)}$$

SINGLETON IF $n = i+j$
EMPTY OTHERWISE

$$= \int_{\mathcal{N} \times \mathcal{N}} \mathcal{N}(i+j, n) \times A_i \times B_j$$

THIS IS DAY'S DEFINITION.