LECTURE SCHEDULE

Monday, April 12, 2021 1:21 PM

4/19 + 4/21 VANESSA + JORGE ON ???

4/26 CHUANYI ON CONDENSED MATTER PHYSICS

4/28 LINUS ON MONADS

5/3 SAMUEL ON MODEL CATS?

5/5 ZHENG ON ALGEBRAIC GEO

MORE ABOUT ∞-CATEGORIES

RECALL AN ∞-CATEGORY IS A QUASI-CATEGORY, I.E A CERTAIN TYPE OF SIMPLICIAL SET

QUESTIONS TO ANSWERED

1) HOW DO WE DEAL WITH ILLDEFINED COMPOSITION OF MORPHISM?
2) WHAT ARE HIGHER MORPHISMS AND WHY ARE THEY INVERTIBLE?
3) HOW DO WE DEFINE LIMITS, COLIMITS AND ADJUNCTIONS

PROPERTIES OF SIMPLICIAL SETS

Δ = CATEGORY OF FINITE ORDERED SETS

A SIMPLICIAL X IS A FUNCTOR

$$\Delta^{op} \longrightarrow \mathcal{S}ets$$

$$\{0, \cdots n\} = [n] \longmapsto X_n$$

X CONSISTS OF SETS X_n FOR $n \geq 0$, AND CERTAIN MAPS BETWEEN THEM

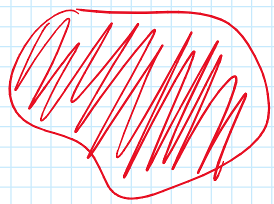
THESE DATA ENABLE US TO BUILD A TOP. SPACE $|X|$, THE GEOMETRIC REALIZATION, WHICH IS A CERTAIN QUOTIENT OF $\coprod_{n \geq 0} X_n \times |\Delta^n|$.

IT HELPS TO THINK OF X AS THE SPACE $|X|$.

DEF A TOP SPACE W IS

DEF A TOP SPACE W IS **CONTRACTIBLE** IF THERE ARE IS A MAP $W \longrightarrow \text{point} \longrightarrow W$ HOMOTOPIC TO THE IDENTITY

CONTRACTIBLE



NOT CONTRACTIBLE

LET Set_Δ BE THE CATEGORY OF SIMPLICIAL SETS

1. **ENRICHMENT** FOR $X, Y \in Set_\Delta$,

$$Set_\Delta(X,Y) = \text{SET OF MORPHISMS } X \longrightarrow Y$$

i.e. MAPS $X_n \longrightarrow Y_n$ COMPATIBLE WITH DATA.

WE CAN THICKEN THIS TO A

SIMPLICIAL SET

$\underline{\mathcal{Set}}_\Delta(X,Y)$ WITH

$$\underline{\mathcal{Set}}_\Delta(X,Y)_n = \mathcal{Set}_\Delta(X \times \Delta^n, Y)$$

$\mathcal{Set}_\Delta$ IS ENRICHED OVER ITSELF. WE WILL SEE LATER THAN FOR $x, y \in X_0$, THE SET OF EDGES $x \to y$ IN X_1 CAN ALSO BE THICKENED INTO A SIMPLICIAL SET $\mathrm{Map}(x,y)$

② KAN MODEL STRUCTURE

② KAN MODEL STRUCTURE ON Set_Δ, DESCRIBED BY QUILLEN IN 1967. A MAP

a) $X \to Y$ IS A COFIBRATION IF $X_n \to Y_n$ IS A MONOMORPHISM OF SETS FOR EACH $n \geq 0$

b) $X \to Y$ IS A WEAK EQUIVALENCE IF $|X| \to |Y|$ IS ONE, MEANING IT INDUCES ISOMORPHISMS OF HOMOTOPY.

THIS MODEL STRUCTURE IS COFIBRANTLY GENERATED

THE FOLLOWING MAPS ARE COFIBRATIONS

$$\partial\Delta^n \xrightarrow{i_n} \Delta^n \quad \text{FOR } n \geq 0$$

THESE ARE THE GENERATING COFIBRATIONS

$$\Lambda^{n-1}_k \to \Delta^n$$

$S^{n-1} \hookrightarrow D^n$ THESE ARE THE GENERATING COFIBRATIONS

<u>THM</u> A MAP $p : X \to Y$ IN $\mathcal{S}et_\Delta$ IS A TRIVIAL FIBRATION IFF

RIGHT LIFTING PROPERTY

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, "i_n"'] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

EXISTS FOR ALL n.

THE FOLLOWING MAPS ARE TRIVIAL (ACYCLIC) COFIBRATIONS

$\Lambda^n_k \xrightarrow{j_{n,k}} \Delta^n$ HORN INCLUSIONS

$\Lambda^n_k \subset \partial\Delta^n$

THESE ARE THE GENERATING TRIVIAL COFIBRATIONS

<u>THEOREM</u> A MAP $p : X \to Y$ IS A FIBRATION IFF

$\Lambda^n_k \longrightarrow X$ LIFTING EXISTS FOR ALL

2

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dotted, "?"] & Y \end{tikzcd}$$

LIFTING EXISTS FOR ALL $n \geq 0$ AND $0 \leq k \leq n$.

THE ABOVE MAPS i_n AND $j_{n,k}$ ARE KNOWN TO DETERMINE THE MODEL STRUCTURE.

IN 2, SUPPOSE $Y = pt$. THEN IF X HAS THIS PROPERTY, WE SAY IT IS A <u>KAN COMPEX</u>.

IN THE KAN MODEL STRUCTURE, ALL OBJECTS ARE COFIBRANT, AND THE FIBRANT OBJECTS ARE THE KAN COMPEXES.

(3) JOYAL MODEL STRUCTURE

IT HAS THE SAME GENERATING COFIBRATIONS AS ABOVE

BUT FEWER GENERATING TRIVIAL COFIBRATIONS, NAMELY THE INNER HORN INCLUSIONS

$$\Lambda^n_k \xrightarrow{j_{n,k}} \Delta^n$$

FOR $0 < k < n$

PROPERTIES

a) SAME COFIBRATIONS AND COFIBRANT OBJECTS AS THE KAN MODEL STRUCTURE

IT HAS THE SAME TRIVIAL

IT HAS THE SAME TRIVIAL FIBRATIONS

b) IT HAS FEWER TRIVIAL COFIBRATIONS, THEREFORE FEWER WEAK EQUIVALENCES MORE FIBRATIONS, AND MORE FIBRANT OBJECTS, NAMELY ALL QUASI-CATS

HANDY FACTS

PROP 2,2 OF GROTH LET $\mathcal{C}$ AND $\mathcal{D}$ BE QUASI-CATS, AND LET K AND M BE SIMP. SETS.

i) THE SIMPLICIAL SET

$$\underline{Set}_{\Delta}(K, \mathcal{C}) =: Fun(K, \mathcal{C})$$

IS A QUASI-CAT

ii) IF A MAP $\mathcal{C} \longrightarrow \mathcal{D}$ IS A JOYAL EQUIV, THEN SO IS $\mathrm{Fun}(K,\mathcal{C}) \to \mathrm{Fun}(K,\mathcal{D})$

iii) IF $K \hookleftarrow M$ IS A JOYAL EQUIV, THEN SO IS

$$\mathrm{Fun}(K,\mathcal{C}) \longrightarrow \mathrm{Fun}(M,\mathcal{C}).$$

WHAT ABOUT ILL DEFINED MORPHISM COMPOSITIONS IN A QUASI-CAT?

<u>THM 1.11</u> (GROTH) FOR A QUASI-CAT $\mathcal{C}$, THE MAP

$$\mathrm{Fun}(\Delta^2,\mathcal{C}) \xrightarrow{p} \mathrm{Fun}(\Lambda^2_1,\mathcal{C})$$

IS A TRIVIAL FIBRATION

IS A TRIVIAL FIBRATION OF SIMPLICIAL SETS.
(THE INCLUSION $\Lambda^2_1 \longrightarrow \Delta^2$ IS A JOYAL EQUIV)

THIS IMPLIES THAT GIVEN

3 $$\begin{tikzcd} x \arrow[r, "f"] \arrow[rr, bend right, "h?"'] & y \arrow[r, "g"] & z \end{tikzcd}$$ IN $\mathcal{C}$,

THIS IS AN ELEMENT OF $\mathrm{Fun}(\Lambda^2_1, \mathcal{C})_0$

THE "SPACE" OF COMPOSITES IS <u>CONTRACTIBLE</u>.

BECAUSE $\mathcal{C}$ IS A QUASI-CAT, THE MAP

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d, "j_{2,1}"'] & \mathcal{C} \\ \Delta^2 \arrow[ur, dotted] & \end{tikzcd}$$

EXTENDS TO Δ^2, GIVING

... TO Δ, GIVING US A "COMPOSITE" h WHICH MAY NOT BE UNIQUE

SINCE p IS A TRIVIAL FIBRATION, THE PREIMAGE OF ANY VERTEX IS CONTRACTIBLE.

THE PREIMAGE OF 3 IS THE "SPACE" OF COMPOSITES gf

ALTERNATE INTERPRETATION:

PULL BACK = FIBER OVER (f,g)

$$\begin{tikzcd}
F_{f,g} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^2, \mathcal{C}) \arrow[d, "p"] \\
\Delta^0 \arrow[r, "(f,g)", "\text{as in } 3"'] & \mathrm{Fun}(\Lambda^2_1, \mathcal{C})
\end{tikzcd}$$

A k-SIMPLEX IN $F_{f,g}$

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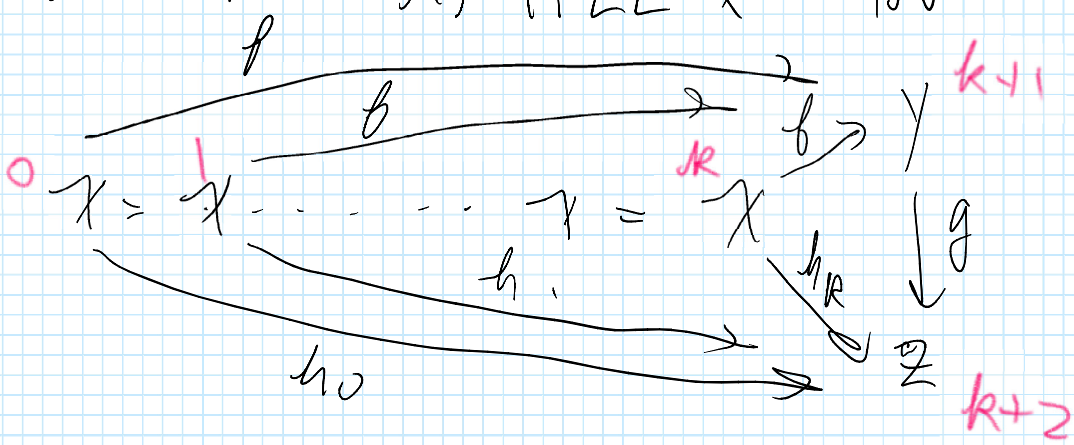

THIS IS A $(k+2)$-SIMPLEX IN $\mathcal{C}$

$i \in [k+2]$

EACH h_i IS A CHOICE OF COMPOSITE. THIS DATA DETERMINES A MAP

$$\Lambda^{k+2}_{k+1} \longrightarrow \mathcal{C}$$

THIS EXTENDS TO Δ^{k+2}

$\rightsquigarrow$ THE FIBER $F_{f,g}$ IS CONTRACTIBLE.

NEXT TIME (MY LAST LECTURE):

HIGHER MORPHISMS, LIMITS, COLIMITS AND (?) ADJUNCTIONS IN $\mathcal{C}$.