MORE ABOUT QUASI-CATEGORIES (Q-CATS)

Wednesday, April 14, 2021 10:50 AM

AKA ∞-CATEGORIES.

RECALL A Q-CAT $\mathcal{C}$ IS A SIMPLICIAL SET SATISFYING THE INNER HORN CONDITION

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & \mathcal{C} \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd} \quad (1)$$

FOR $0<k<n$ (IT NEED NOT EXTEND FOR $k=0$ AND $k=n$)

WHY IS THIS AN $(\infty,1)$ CATEGORY?

AN ORDINARY SMALL CATEGORY $\mathcal{C}$ HAS A NERVE $N(\mathcal{C}) \in Set_\Delta$ = CAT OF SIMP SETS

PROP A SIMPLICIAL SET X IS THE NERVE OF A SMALL CAT $\iff$ IT IS A Q-CAT WITH UNIQUE LIFTING IN (1). THE OBJECT SET IS X_0 AND THE MORPHISM SET IS X_1

IN A Q-CAT, COMPOSITION OF MORPHISMS IS NOT WELL DEFINED. INSTEAD

WE GET A CONTRACTIBLE SPACE OF CHOICES.

TOWARD LIMITS AND COLIMITS IN Q-CATS:

RECALL A TERMINAL OBJECT $*$ IN AN ORDINARY CATEGORY $\mathcal{C}$ IS ONE THAT ADMITS A UNIQUE MORPHISM FROM ANY OBJECT X.

IN AN ∞-CAT $\mathcal{C}$, WE WANT THE SIMP SET $\mathcal{C}(X, *)$ TO BE CONTRACTIBLE FOR ALL X.

RELATION BETWEEN LIMITS AND TERMINAL OBJECTS IN ORDINARY CATEGORY

EXAMPLE: EQUALIZER OF TWO MORPHISMS

$$X \underset{b}{\overset{a}{\rightrightarrows}} Y$$

2

IT IS AN OBJECT W WITH A MAP TO $W \xrightarrow{f} X$ S.T. $af = bf$ AND FOR OTHER

... TO $W \to X$ S.T.

$af = bf$ AND FOR OTHER

$W' \xrightarrow{f'} X \rightrightarrows Y$ WITH $af' = bf'$, $\exists!$

$W' \xrightarrow{g} W$ WITH $fg = f'$.

CONSIDER THE CATEGORY OF ALL SUCH PAIRS (W', f') AS ABOVE

WITH MORPHISMS

$$\begin{tikzcd}
W'_1 \arrow[dr, "f'_1"] \arrow[dd, "g"'] & & \\
& X \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & Y \\
W'_2 \arrow[ur, "f_2"'] & &
\end{tikzcd}$$

CALL THIS CATEGORY

$\mathcal{C}_{/X \rightrightarrows Y}$ = OBJECTS IN $\mathcal{C}$ LYING OVER THE DIAGRAM

$$X \overset{a}{\underset{b}{\rightrightarrows}} Y$$

A LIMIT W (IF IT EXISTS) IS A TERMINAL OBJECT IN THIS CATEGORY.

FOR A Q-CAT $\mathcal{C}$ WE CAN DEFINE $\mathcal{C}_{/X \rightrightarrows Y}$ IN THE SAME WAY AND SHOW IT IS ALSO A Q-CAT. A LIMIT (IF IT EXISTS) IS A TERMINAL OBJECT.

<u>DEF</u> A Q-CAT IS COMPLETE (COCOMPLETE) IF ALL SMALL LIMITS (COLIMITS) EXIST.

HOMOTOPY IN A Q-CAT BETWEEN TWO MORPHISMS

$$X \overset{f}{\underset{g}{\rightrightarrows}} Y \qquad f \simeq g$$

ARE <u>HOMOTOPIC</u> IN $\mathcal{C}$ IF THERE IS A 2-SIMPLEX

$1_X \nearrow X \searrow g$, θ

THIS IS ALSO A 2-MORPHISM

$$
\begin{tikzcd}
& X \arrow[dr, "g"] & \\
X \arrow[ur, "1_X"] \arrow[rr, "f"'] \arrow[rr, phantom, shift left=3, "\theta_1"] & & Y
\end{tikzcd}
$$

THIS IS ALSO A 2-MORPHISM $f \Rightarrow g$ IN $\mathcal{C}$

CAN SHOW THIS LEADS TO AN EQUIVALENCE RELATION ON MORPHISM $X \to Y$.

PROOF OF SYMMETRIC

NEED TO SHOW THAT IF θ_1 IS A 2-SIMPLEX, SO IS

$$
\begin{tikzcd}
& X \arrow[dr, "g"] & \\
X \arrow[ur, "1_X"] \arrow[rr, "f"'] \arrow[rr, phantom, shift left=3, "\theta_2"] & & Y
\end{tikzcd}
$$

CONSIDER A POSSIBLE 3-SIMPLEX

$$
\begin{tikzcd}
X \arrow[dd, equal] \arrow[dr, equal] \arrow[drr, bend left=40, "f"] & & \\
& X \arrow[r, "g"] \arrow[ur, phantom, "\theta_1"] \arrow[dr, phantom, "\theta_1^{-1}"] & Y \\
X \arrow[ur, equal] \arrow[urr, bend right=40, "f"'] & &
\end{tikzcd}
$$

THE DIAGRAM DEFINES A HORN

$$\Lambda^3_2 \longrightarrow \mathcal{C}$$

WITH FACES

$$(\theta_1^{-1}, \theta_1, -, X)$$

$(\theta_1, \theta_1, -, x)$

THIS MAP EXTENDS TO $J : \Delta^3 \to \mathcal{C}$

WITH $d_2 J =$

$$\begin{tikzcd} \overset{0}{X} \arrow[rr, "f"] \arrow[d, equal] & & Y \; 3 \\ \underset{1}{X} \arrow[urr, "f"'] \arrow[urr, phantom, "K_f", shift left=2] & & \end{tikzcd}$$

THIS IS IMPLIES HOMOTOPY IS SYMMETRIC.

<u>REMARK</u> WE COULD ALSO DEFINE HOMOTOPY $f \simeq g$ WITH

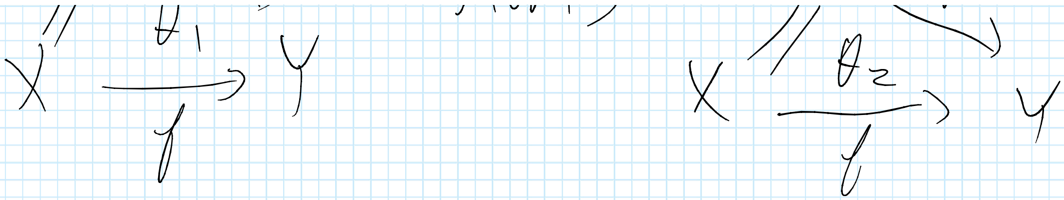
$$\begin{tikzcd} & Y \; 1 \\ 0 \; X \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & Y \; 2 \arrow[uu, equal] \end{tikzcd}$$

CAN SHOW THIS LEADS TO THE SAME EQUIVALENCE RELATION.

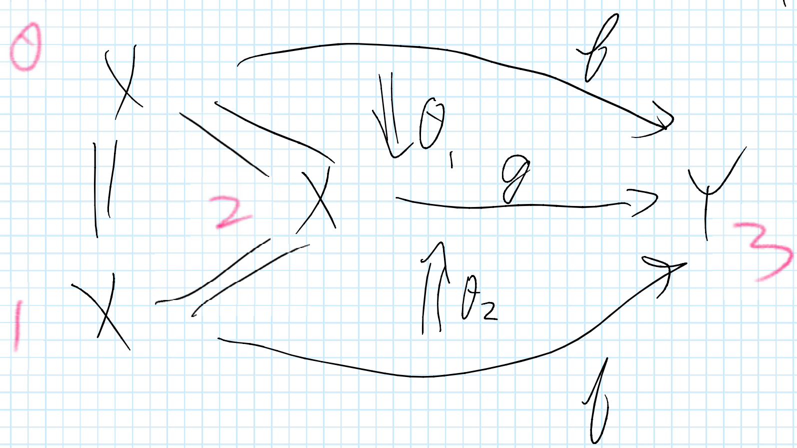
WHAT IS A 3-MORPHISM IN A Q-CAT?

SUPPOSE WE HAVE TWO 2-MORPHISM

$$\begin{tikzcd} & X \arrow[dr, "g"] & \\ X \arrow[ur, equal] \arrow[rr] \arrow[rr, phantom, "\theta_1", shift left=2] & & Y \end{tikzcd} \quad \text{AND} \quad \begin{tikzcd} & X \arrow[dr, "g"] & \\ X \arrow[ur, equal] \arrow[rr] \arrow[rr, phantom, "\theta_2", shift left=2] & & Y \end{tikzcd}$$



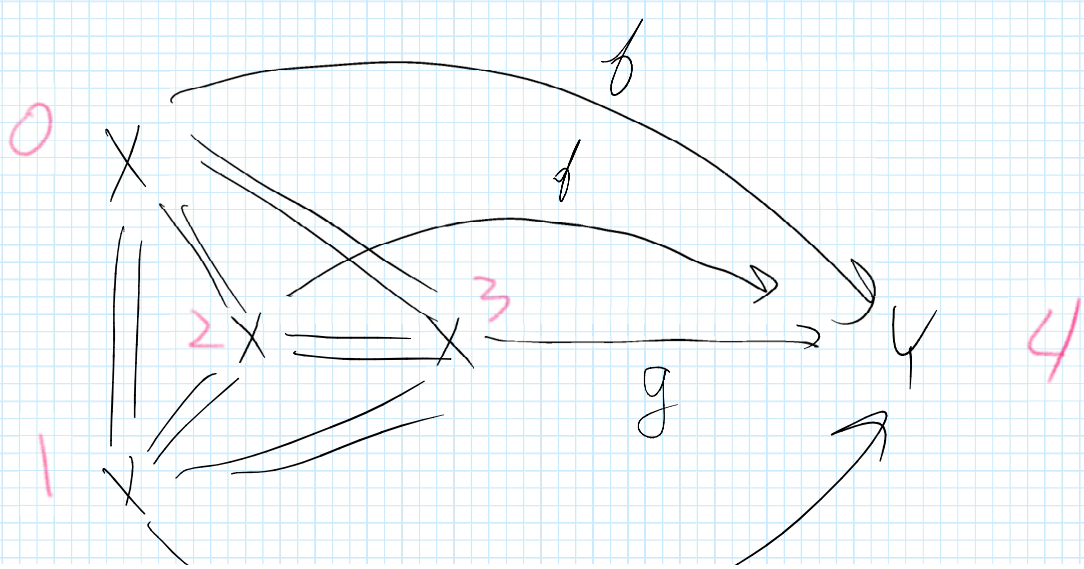
A 3-MORPHISM $\varsigma_1 : \theta_1 \Rrightarrow \theta_2$

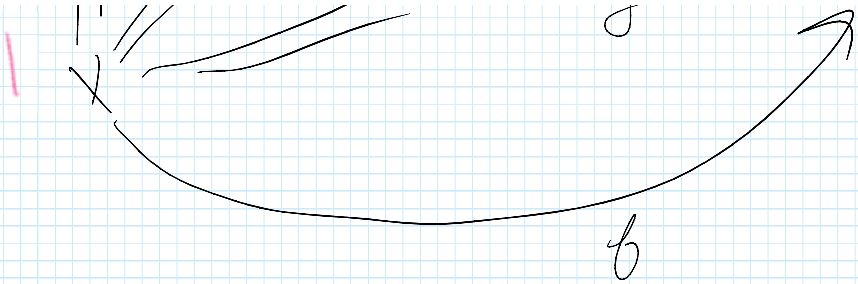


$\theta_1, \theta_2 : f \Rightarrow g$

WE CAN DEFINE HIGHER MORPHISMS IN A SIMILAR WAY.

FOR A 4-MORPHISM WE NEED A 4-SIMPLEX





THE VERTICES 0, 2, 3, 4
GIVES A 3-MORPHISM $S_1 : \theta_1 \Rrightarrow \theta_2$

THE VERTICES 0, 1, 3, 4
DEFINE ANOTHER 3-MORPHISM
$$S_2 : \theta_1 \Rrightarrow \theta_2$$

THE WHOLE PICTURE REPRESENT
A 4-MORPHISM $S_1 \Rrightarrow S_2$

NOTE THERE ARE 6 2-FACES,
WITH 4 AS A VERTEX.
EACH REPRESENTS A 2-MORPHISM

0 1 4 $\rightsquigarrow$ $1_f : f \Rightarrow f$

0 2 4 SAME

0 3 4 $\theta_1 : f \Rightarrow g$

1 2 4 $1_f : f \Rightarrow f$???

1 3 4 $\theta_2 : f \Rightarrow g$

1 3 4 $\quad \theta_2 : f \Rightarrow g$ . . .

2 3 4 $\quad \theta_3 : f \Rightarrow g$

THERE FOUR 3-FACES WITH
4 AS A VERTEX
EACH REPRESENTS A 3-MORPHISM

1 2 3 4
0 2 3 4
0 1 3 4
0 1 2 4

WHAT ARE THEY ?

THEY ARE EACH ONE OF THE FOLLOWING

$$S_1, S_2 : \theta_1 \Rrightarrow \theta_2$$

$$1_{\theta_1} : \theta_1 \Rrightarrow \theta_1$$

$$1_{\theta_2} : \theta_2 \Rrightarrow \theta_2$$