Monday, April 5, 2021 1:00 PM

DEF CAT (Cat) DENOTES THE CATEGORY OF (SMALL) CATEGORIES

OBJECTS: (SMALL) CATEGORIES

MORPHISMS: FUNCTORS

WE ALSO HAVE NATURAL TRANSFORMATIONS, WHICH "MORPHISMS BETWEEN MORPHISMS" OR 2-MORPHISMS, AKA 2-CELLS

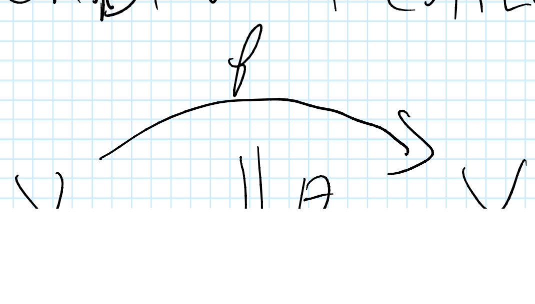
IN THIS JARGON, ORDINARY MORPHISMS ARE 1-MORPHISMS OR 1-CELLS AND OBJECTS ARE 0-CELLS.

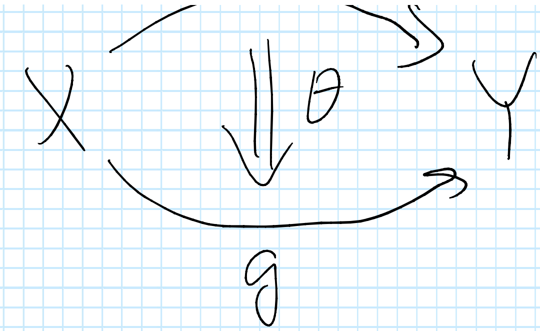
DEF A STRICT <u>2-CATEGORY</u> $\mathcal{C}$ CONSISTS

a) A COLLECTION OF OBJECTS

b) MORPHISMS BETWEEN THEM AS IN AN ORDINARY CATEGORY

c) 2-MORPHISMS



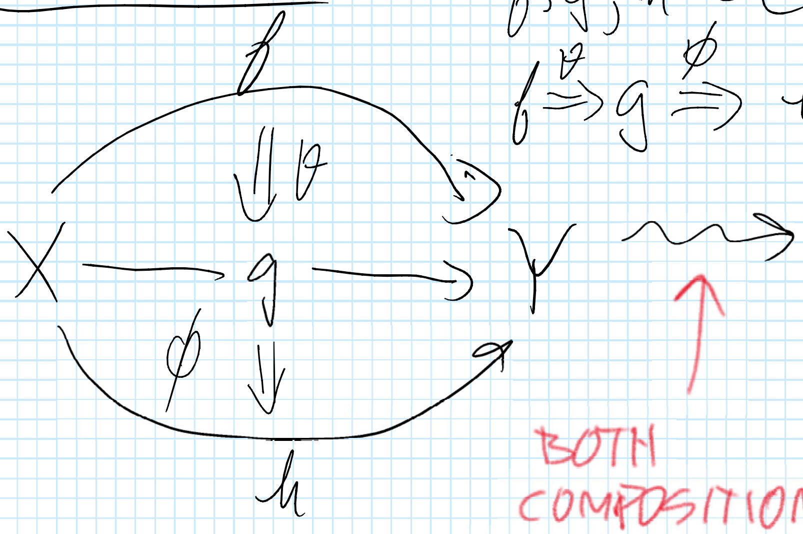


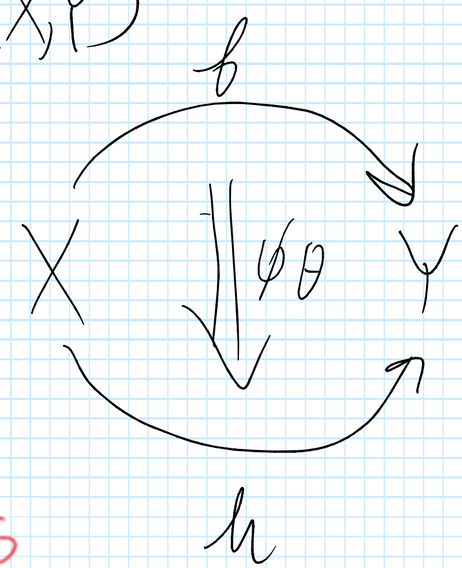
WHICH CAN BE COMPOSED IN 2 WAYS

VERTICAL

$f, g, h \in \mathcal{C}(X,Y)$

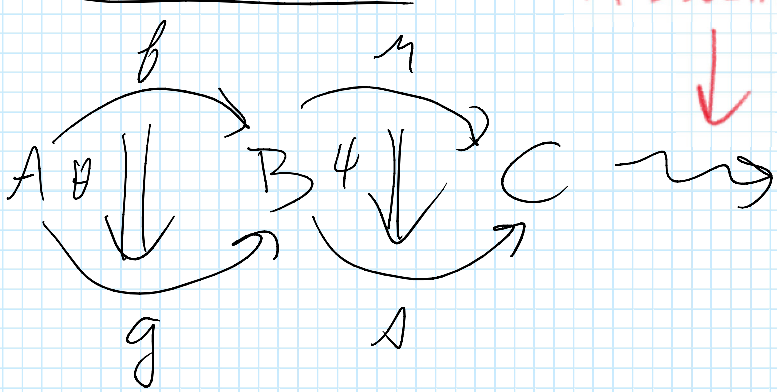
$f \overset{\theta}{\Rightarrow} g \overset{\phi}{\Rightarrow} h$

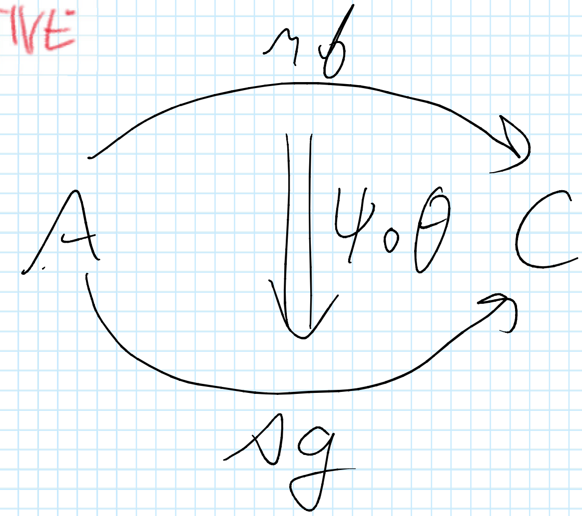




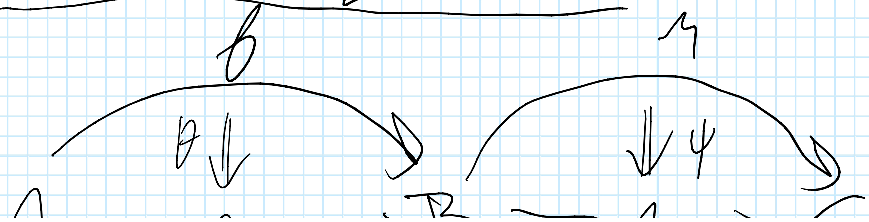
BOTH COMPOSITIONS ARE ASSOCIATIVE

HORIZONTAL

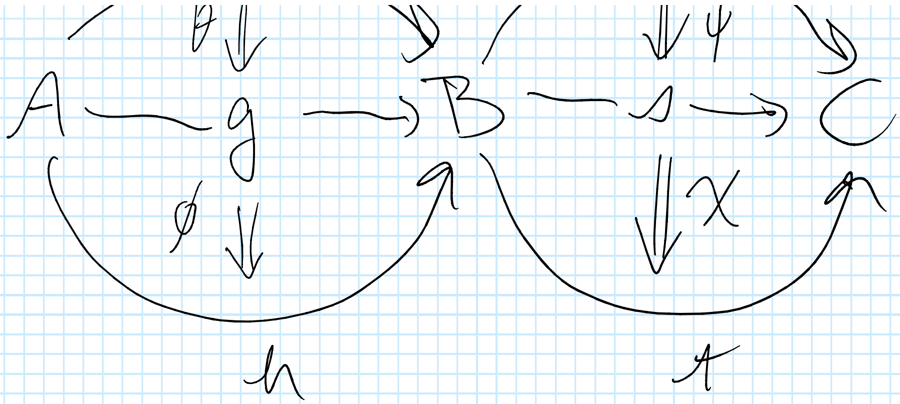




INTERCHANGE LAW



$(\chi\psi)\circ(\phi\theta)$



STRICT

$(\chi \circ \phi)\ (\psi \circ \theta)$

AS 2-MORPHISMS $sf \Rightarrow th$

THIS DEFINITION IS EVIL
WE COULD REQUIRE THEM TO BE NATURALLY ISOMORPHIC.

REMARK: WE COULD DEFINE HIGHER (THAN 2-) MORPHISMS AND REQUIRE THEM TO SATISFY VARIOUS CONDITIONS.

ALTERNATE INTERPRETATION OF A STRICT 2-CATEGORY:
IT IS A CATEGORY "ENRICHED OVER CAT OR $\mathcal{C}at$."
THIS MEANS THAT FOR OBJECTS A, B IN $\mathcal{C}$

OBJECTS A, B IN $\mathcal{C}$, $\mathcal{C}(A,B)$ IS AN OBJECT IN CAT OR $\mathcal{C}at$.

WE NEED A MONOIDAL STRUCTURE ON CAT or $\mathcal{C}at$. GIVEN CATEGORIES $\mathcal{C}_1$ AND $\mathcal{C}_2$, WE CAN DEFINE $\mathcal{C}_1 \times \mathcal{C}_2$, WHOSE OBJECTS ARE PAIRS (X_1, X_2) WITH $X_1 \in \mathcal{C}_1$
$X_2 \in \mathcal{C}_2$

AND MORPHISMS $(X_1, X_2) \to (Y_1, Y_2)$ ARE . . . . .

THE UNIT IS THE TRIVIAL CATEGORY $\mathbb{1}$, WITH ONE OBJECT AND ONE MORPHISM.

REMARK AN ORDINARY CATEGORY $\mathcal{C}$ IS A 2-CATEGORY IN WHICH

IS A 2-CATEGORY IN WHICH THE ONLY 2-MORPHISMS ARE IDENTITIES. THE SET $\mathcal{C}(A,B)$ CAN BE THOUGHT OF AS A CATEGORY WHOSE OBJECTS FORM THE SET $\mathcal{C}(A,B)$, AND ALL MORPHISMS ARE IDENTITIES

THIS IS THE <u>DISCRETE CATEGORY</u> FOR THE SET $\mathcal{C}(A,B)$.

ORDINARY CATEGORIES ARE TO 2-CATEGORIES AS DISCRETE CATEGORIES (SETS) ARE TO SMALL CATEGORIES

<u>PROGRAM</u> DEFINE AN $(n+1)$-CATEGORY TO BE A CATEGORY ENRICHED OVER (PREVIOUSLY DEFINED) n-CATEGORIES.

ROUGH DEFINITION: AN (∞, n)-CATEGORY IS AN ∞-CATEGORY IN WHICH ALL k-MORPHISMS FOR $k > n$ ARE INVERTIBLE.

OBJECTS OF INTEREST ARE $(\infty, 1)$-CATEGORIES, COMMONLY KNOWN SIMPLY AS ∞-CATEGORIES.

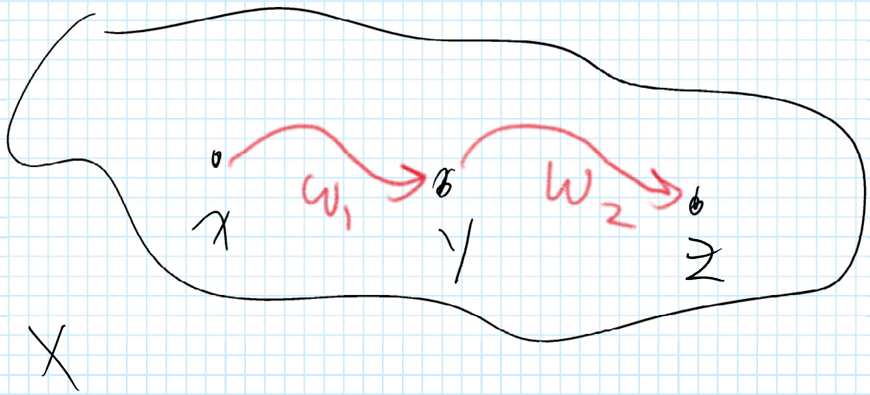
WANT TO DEFINE QUASI-CATEGORIES, WHICH ARE EXAMPLES OF $(\infty, 1)$-CATEGORIES. THEY ARE CERTAIN SIMPLICIAL SETS.

GUIDING EXAMPLE OF AN $(\infty, 0)$-CATEGORY, AKA ∞-GROUPOID. LET X BE A TOPOLOGICAL SPACE. ITS <u>FUNDAMENTAL GROUPOID</u>

$\Pi(X)$ IS THE CATEGORY IN WHICH

OBJECTS ARE POINTS $x \in X$

MORPHISMS $x \to y$ "ARE" PATHS FROM x TO y, I.E. CONTINUOUS MAPS $I=[0,1] \xrightarrow{\omega} X$ WITH $\omega(0)=x$ AND $\omega(1)=y$.



NEED TO PRECOMPOSE WITH $[0,1] \to [0,2]$

ASSOCIATIVITY IS A PROBLEM.

<u>SOLUTION</u> DEFINE AN EQUIVALENCE RELATION (HOMOTOPY) AMONG PATHS $x \to y$, AND DEFINE A MORPHISM

$x \to y$, AND DEFINE A MORPHISM $x \to y$ IN $\Pi(X)$ TO BE AN EQUIVALENCE (HOMOTOPY) CLASS OF PATHS $x \to y$

DEF TWO MAPS $f, g : W \to X$ ARE <u>HOMOTOPIC</u> IF THERE IS A MAP $h : I \times W \to X$

WITH $h(0, x) = f(x)$

AND $h(1, x) = g(x)$

"f CAN BE CONTINUOUSLY DEFORMED TO g"

THIS I (HOMOTOPY PARAMETER) DIFFERS FROM THE I ABOVE, THE PATH PARAMETER.

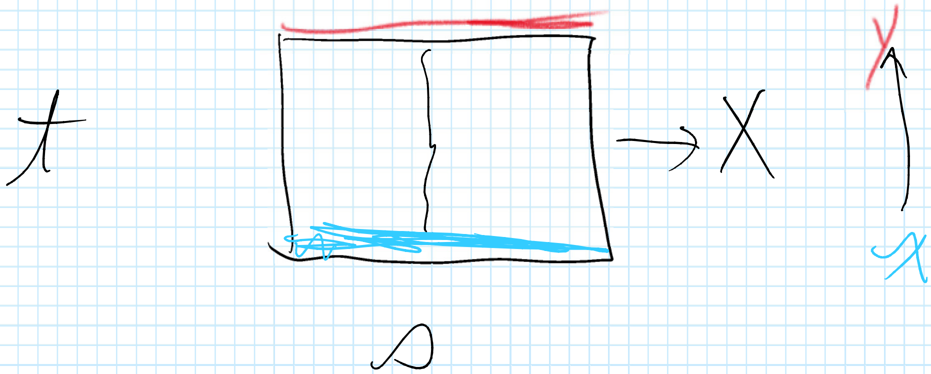
IN THE CASE OF PATHS ($W = I$) WE HAVE $I \times I \xrightarrow{h} X$

WE HAVE $I \times I \xrightarrow{h} X$

WE WANT $h(s,0) = f(0) = g(0) = x$

$h(s,1) = f(1) = g(1) = y$

FOR ALL $0 \leq s \leq 1$



MORPHISMS $x \to y$ IN THE CATEGORY $\pi(X)$ ARE HOMOTOPY CLASSES OF PATHS $x \to y$, RATHER THAN THE PATHS THEMSELVES.

THIS MAKES COMPOSITION ASSOCIATIVE AS DESIRED.

A n-MORPHISM IS AN

EQUIVALENCE CLASS OF SUITABLE MAPS $I^k \to X$

REMARKS

① THE IDENTITY MAP IN $\pi(X)(x,x)$ IS THE CLASS OF THE CONSTANT PATH $x \to x$

② THE INVERSE OF THE CLASS OF A PATH ω FROM x TO y IS $\bar{\omega}(t) = \omega(1-t)$.

EVERY MORPHISM IN $\pi(X)$ IS INVERTIBLE.

THE GROTHENDIECK HOMOTOPY HYPOTHESIS: EVERY $(\infty,0)$-CATEGORY "LOOKS LIKE" $\pi(X)$ FOR SOME SPACE X.

EVERY CATEGORY IS AN $(\infty, 1)$

EVERY CATEGORY IS AN $(\infty, 1)$-CATEGORY IN WHICH EACH k-MORPHISM FOR $k \geq 2$ IS AN IDENTITY AND THEREFORE INVERTIBLE.