Algebraic theories

Charlotte Aten

University of Rochester

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Some history

- A 1935 paper of Garrett Birkhoff was the starting point for a new discipline within algebra, called either «universal algebra» or «general algebra», which deals with those properties common to all algebraic structures, such as groups, rings, Boolean algebras, Lie algebras, and lattices.
- Birkhoff used lattice-theoretic ideas in his paper. In 1940 he published a book on lattice theory.
- Øystein Ore referred to lattices as «structures» and led a short-lived program during the 1930s where lattices were hailed as the single unifying concept for all of mathematics.
- During this period Saunders Mac Lane studied algebra under Ore's advisement. Mac Lane went on to become one of the founders of category theory and coauthored an influential algebra textbook with Birkhoff.

Some history

- Over the next few decades universal algebra and algebraic topology grew into decidedly separate disciplines.
- By the 1960s Bill Lawvere was a student of Samuel Eilenberg.
- Although most of Eilenberg's students were algebraic topologists, Lawvere wrote a thesis on universal algebra.
- Eilenberg famously did not even read Lawvere's thesis before Lawvere was awarded his Ph.D.
- However, Eilenberg did finally read it in preparation for his lectures on «Universal algebra and the theory of automata» at the 1967 AMS summer meeting in Toronto.
- This talk is on the category theoretic treatment of universal algebra, which was the topic of Lawvere's thesis.

Talk outline

- Algebras in universal algebra
- Clones
- Equational theories
- Algebraic theories
- Monads and algebraic theories

Algebras in universal algebra

Operations are rules for combining elements of a set together to obtain another element of the same set.

Definition (Operation, arity)

Given a set A and some $n \in \mathbb{W}$ we refer to a function $f\colon A^n \to A$ as an *n-ary operation* on A. When f is an n-ary operation on A we say that f has *arity* n.

Algebras in universal algebra

Algebras are sets with an indexed sequence of operations.

Definition (Algebra)

An *algebra* (A, F) consists of a set A and a sequence $F = \{f_i\}_{i \in I}$ of operations on A, indexed by some set I.

Algebras in universal algebra

- Given an algebra $\mathbf{A} := (A, \{f_i\}_{i \in I})$ we define a map $\rho\colon I \to \mathbb{W}$ where $\rho(i) := n$ when $f_i\colon A^n \to A$ is an n-ary operation on A.
- This map $\rho\colon I \to \mathbb{W}$ is called the *similarity type* of $\mathbf{A}$.
- When two algebras $\mathbf{A} := (A, F)$ and $\mathbf{B} := (B, G)$ have the same similarity type $\rho\colon I \to \mathbb{W}$ we say that $\mathbf{A}$ and $\mathbf{B}$ are *similar algebras*.

Algebras in universal algebra

Definition (Homomorphism)

Given algebras $\mathbf{A} := (A, F)$ and $\mathbf{B} := (B, G)$ of the same similarity type $\rho\colon I \to \mathbb{W}$ we say that a function $h\colon A \to B$ is a *homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ when for each $i \in I$ and all $a_1, \ldots, a_{\rho(i)} \in A$ we have that

$$h(f_i(a_1, \ldots, a_{\rho(i)})) = g_i(h(a_1), \ldots, h(a_{\rho(i)})).$$

Algebras in universal algebra

- The class Alg_ρ of all algebras of signature ρ can be taken as the objects of a category $\mathbf{Alg}_\rho$ whose morphisms are algebra homomorphisms.
- The category $\mathbf{Alg}_\rho$ has some particularly well-behaved full subcategories: We say that a class of algebras $\mathcal{K} \subset \mathrm{Alg}_\rho$ is a *variety* when $\mathcal{K}$ is closed under taking homomorphic images, subalgebras, and products.
- Examples of varieties include (abelian/quasi/semi) groups, (unital/commutative/Lie) rings, and (semi/distributive/modular) lattices.
- Fields do not form a variety.

Clones

- We will need to keep track of all functions which can be built using the basic operations of an algebra.
- Given $n \in \mathbb{W}$ and a set A we define $\mathrm{Op}_n(A) := A^{(A^n)}$.
- Given $n, k \in \mathbb{W}$, $f \in \mathrm{Op}_n(A)$, and $g_1, \ldots, g_n \in \mathrm{Op}_k(A)$ the *generalized composite*

$$f[g_1, \ldots, g_n] \colon A^k \to A$$

is given by

$$f[g_1, \ldots, g_n](x_1, \ldots, x_k) := f(g_1(x), \ldots, g_n(x)).$$

- Note that $\mathrm{Op}_n(A)$ contains all the coordinate projections p_k^n where

$$p_k^n(x_1, \ldots, x_n) := x_k.$$

Clones

Definition (Clone)

Given a nonempty set A we say that $C \subset \mathrm{Op}(A) := \bigcup_{n \in \mathbb{W}} \mathrm{Op}_n(A)$ is a *clone* when C is closed under generalized composition and contains all the coordinate projection operations.

- The largest clone on A is $\mathrm{Op}(A)$ itself.
- The smallest clone on A is $\mathrm{Proj}(A) := \{ p_k^n \mid 1 \leq k \leq n \in \mathbb{W} \}$.
- Clones are examples of operads whose operation spaces are discrete.

Clones

Definition (Term)

Given a similarity type $\rho\colon I \to \mathbb{W}$, a set of variables X, and a set $F := \{f_i\}_{i \in I}$ which we think of as abstract basic operation symbols, a *term* in the language of ρ in the variables X is an element of the set $T_\rho(X) := \bigcup_{n \in \mathbb{W}} T_n$ where

$$T_0 := X \cup \{\, f_i \mid \rho(i) = 0 \,\}$$

and for $n \in \mathbb{W}$ we set

$$T_{n+1} := T_n \cup \{\, f_i[t_1, \ldots, t_k] \mid i \in I,\ k = \rho(i), \text{ and } t_1, \ldots, t_k \in T_n \,\}\,.$$

Clones

- That is, $T_\rho(X)$ consists of all valid formal composites of the basic operation symbols $\{f_i\}_{i \in I}$ whose arities are given by ρ with variable arguments coming from the set X.
- Given an algebra $\mathbf{A}$ of signature ρ and a term $t(x_1, \ldots, x_n) \in T_\rho(\{x_1, \ldots, x_n\})$ we define the *term operation*

$$t^{\mathbf{A}} \colon A^n \to A$$

  by interpreting all the operation symbols appearing in t as actual basic operations of $\mathbf{A}$ in the obvious way.
- For example, if ρ is the usual signature for groups then $(xy)(x^{-1}y^{-1})$ is a term in the variables $\{x, y\}$ whereas there exists an actual commutator term operation on the symmetric group $\mathbf{S}_3$ which is a binary operation on S_3.

Clones

- Each algebra **A** has a corresponding clone of term operations, which is
$$\mathrm{Clo}(\mathbf{A}) := \bigcup_{n \in \mathbb{W}} \mathrm{Clo}_n(\mathbf{A})$$
where
$$\mathrm{Clo}_n(\mathbf{A}) := \left\{ t^{\mathbf{A}} \;\middle|\; t \in T_\rho(\{x_1, \ldots, x_n\}) \right\}.$$
- This is to say that $\mathrm{Clo}(\mathbf{A})$ consists of all the operations on A which can be built up using the basic operations of A and (implicitly) projections.
- Another way of saying this is that $\mathrm{Clo}(\mathbf{A})$ is the smallest clone in the lattice of clones on A which contains the basic operations of **A**.

Equational theories

Definition (Identity)

To each ordered pair $(t_1, t_2) \in T_\rho(\{x_1, \ldots, x_n\})$ we associate a proposition, called an *identity* in the language of ρ, which is

$$(\forall x_1, \ldots, x_n)(t_1(x_1, \ldots, x_n) = t_2(x_1, \ldots, x_n)).$$

We usually write

$$t_1(x_1, \ldots, x_n) \approx t_2(x_1, \ldots, x_n)$$

or even $t_1 \approx t_2$ instead.

- Given a ρ-algebra $\mathbf{A}$ and an identity ϵ we say that $\mathbf{A}$ *models* $\epsilon = t_1 \approx t_2$ and write $\mathbf{A} \models \epsilon$ when

$$(\forall x_1, \ldots, x_n \in A)(t_1^{\mathbf{A}}(x_1, \ldots, x_n) = t_2^{\mathbf{A}}(x_1, \ldots, x_n)).$$

Equational theories

Definition (Equational theory)

Given a class $\mathcal{K}$ of algebras of signature ρ we say refer to a set of equations of the form

$$\mathsf{Id}(\mathcal{K}) := \{\, \epsilon \mid (\forall \mathbf{A} \in \mathcal{K})(\mathbf{A} \models \epsilon) \,\}$$

as an *equational theory*.

- For example, when $\mathcal{K}$ is the class of all groups we have that $\mathsf{Id}(\mathcal{K})$ contains identities like $(xy)z^{-1} \approx x(yz^{-1})$ and $x(ex^{-1}) \approx e$ but does not contain $xy \approx yx$ or $x(xx) \approx x$.

Equational theories

Definition (Equational class)

We say that a class $\mathcal{K}$ of ρ-algebras is *equational* when there exists a set of identities $\Sigma \subset (T_\rho(X))^2$ such that

$$\mathcal{K} = \mathrm{Mod}(\Sigma) := \{ \mathbf{A} \mid \mathbf{A} \models \Sigma \} .$$

- All of the varieties that we have mentioned are equational classes by definition.
- It is trivial to see that any equational class is a variety.
- The converse is not trivial to prove, but it is true.

Theorem (Birkhoff)

A class of ρ-algebras is equational if and only if it is a variety.

Equational theories

- Recall that any variety of algebras $\mathcal{V}$ is a full subcategory of $\mathbf{Alg}_\rho$.
- There is a forgetful functor $U\colon \mathcal{V} \to \mathbf{Set}$ given by $U(A, F) := A$.
- This functor has a left adjoint $H\colon \mathbf{Set} \to \mathcal{V}$ which assigns to a set X the free algebra $\mathbf{H}(X)$ on generators X in the variety $\mathcal{V}$.
- This free algebra functor can be constructed quite explicitly.
- We can take
$$\mathbf{H}(X) := \mathbf{T}_\rho(X)/\theta_{\mathcal{V}}$$
where $\theta_{\mathcal{V}}$ is the congruence obtained by identifying terms t_1 and t_2 when $\mathcal{V} \models t_1 \approx t_2$.

Algebraic theories

Definition (Algebraic theory)

An *algebraic theory* is a small category $\mathcal{T}$ with finite products.

Definition (Algebra of an algebraic theory)

Given an algebraic theory $\mathcal{T}$ an *algebra* of $\mathcal{T}$ is a functor $\mathbf{A}\colon \mathcal{T} \to \mathbf{Set}$ which preserves finite products.

Definition (Category of algebras of an algebraic theory)

Given an algebraic theory $\mathcal{T}$ we define $\mathbf{Alg}(\mathcal{T})$ to be the full subcategory of $\mathbf{Set}^{\mathcal{T}}$ whose objects are algebras of $\mathcal{T}$.

Algebraic theories

Definition (Algebraic category)

We refer to a category which is equivalent to $\mathbf{Alg}(\mathcal{T})$ as an *algebraic category*.

- One example of an algebraic category is **Set**.
- We can take $\mathbf{Set} \cong \mathbf{Alg}(\mathcal{W})$ where $\mathcal{W}$ is the full subcategory of $\mathbf{Set}^{\mathrm{op}}$ whose objects are the sets $[n] := \{0, 1, \ldots, n-1\}$ for $n \in \mathbb{W}$.
- Note that if $\mathbf{A} \colon \mathcal{W} \to \mathbf{Set}$ is an algebra of $\mathcal{W}$ then

$$\mathbf{A}[n] = \mathbf{A}([1] \times \cdots \times [1]) = \mathbf{A}[1] \times \cdots \times \mathbf{A}[1]$$

so $\mathbf{A}$ is determined by $\mathbf{A}[1]$.

Algebraic theories

Definition (Algebraic category)

We refer to a category which is equivalent to $\mathbf{Alg}(\mathcal{T})$ as an *algebraic category*.

- The functor $\mathbf{A}$ is determined by $A := \mathbf{A}[1]$.
- A morphism $[n] \to [1]$ in $\mathbf{Set}^{\mathrm{op}}$ is a function from $[1] \to [n]$.
- Given a morphism $f\colon [n] \to [1]$ in $\mathbf{Set}^{\mathrm{op}}$ we have

$$\mathbf{A}f\colon A^n \to A.$$

- We have that $\mathrm{Proj}_n(A) = \{\, \mathbf{A}f \mid f\colon [n] \to [1] \text{ in } \mathbf{Set}^{\mathrm{op}} \,\}$.

Algebraic theories

- For a different example of an algebraic category we can take **Grp**, the category of groups.
- In this case we have $\mathbf{Grp} \cong \mathbf{Alg}(\mathcal{G})$ where $\mathcal{G}$ has the sets $[n]$ for $n \in \mathbb{W}$ as objects and we define

$$\mathcal{G}([n],[1]) \coloneqq H(\{x_1,\ldots,x_n\})$$

  where $H\colon \mathbf{Set} \to \mathbf{Grp}$ is the free group functor.
- A functor $\mathbf{A}\colon \mathcal{G} \to \mathbf{Set}$ selects an object $\mathbf{A}[1]$ (the set of elements of the group $\mathbf{A}$) for which we have that

$$\mathrm{Clo}_n(\mathbf{A}) = \{\, \mathbf{A}f \mid f\colon [n] \to [1] \text{ in } \mathcal{G} \,\}\,.$$

Monads and algebraic theories

Definition (Monoid object)

A *monoid* (M, m, e) in a category $\mathscr{C}$ is an object M along with morphisms $m\colon M^2 \to M$ and $e\colon 1 \to M$ such that m is an associative operation with unit e, in the sense that the diagrams below commute.

$$
\begin{tikzcd}
M \times M \times M \arrow[r, "m \times \mathrm{id}_M"] \arrow[d, "\mathrm{id}_M \times M"'] & M \times M \arrow[d, "m"] \\
M \times M \arrow[r, "m"] & M
\end{tikzcd}
$$

$$
\begin{tikzcd}
& M & \\
M = 1 \times M \arrow[ur, "\mathrm{id}_M"] \arrow[r, "e \times \mathrm{id}_M"'] & M \times M \arrow[u, "m"] & M \times 1 = M \arrow[ul, "\mathrm{id}_M"'] \arrow[l, "\mathrm{id}_M \times e"]
\end{tikzcd}
$$

Monads and algebraic theories

Definition (Monad)

A *monad* $\mathbb{M} := (M, \mu, \eta)$ on a category $\mathscr{C}$ is an endofunctor $M\colon \mathscr{C} \to \mathscr{C}$ along with natural transformations $\mu\colon MM \to M$ and $\eta\colon \mathrm{id}_{\mathscr{C}} \to M$ such that μ is an associative operation with unit η, in the sense that the diagrams below commute.

$$\begin{tikzcd}
MMM \arrow[r, "\mu M"] \arrow[d, "M\mu"'] & MM \arrow[d, "\mu"] \\
MM \arrow[r, "\mu"] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
& M & \\
M \arrow[ur, "M"] \arrow[r, "\eta M"'] & MM \arrow[u, "\mu"] & M \arrow[ul, "M"'] \arrow[l, "M\eta"]
\end{tikzcd}$$

Monads and algebraic theories

- Any variety of algebras $\mathcal{V}$ can be viewed as an algebraic theory analogous to the example with groups which we gave before.
- Let $M\colon \mathbf{Set} \to \mathbf{Set}$ denote the monad induced by the free-forgetful adjunction for $\mathcal{V}$. That is, take $M = UH$.
- We can thus obtain from any variety of algebras both an algebraic theory and a monad.
- It is possible to define equations and equational categories (analogues of equational classes in universal algebra) for algebraic theories.
- It turns out that equational categories are up to concrete isomorphism precisely the categories $\mathbf{Set}^{\mathbb{M}}$ of Eilenberg-Moore algebras for finitary monads $\mathbb{M}$ on $\mathbf{Set}$.

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