Note Title 10/15/2012

Lemma G = finite gp

$V = V_1 + V_2$ reps of G

$d_i = \dim V_i \quad i = 1, 2$

$X = S(V_1) = S^{d_1 - 1}$

$Y = S(V_2) = S^{d_2 - 1}$

$W = X * Y = S(V) = S^{d_1 + d_2 - 1}$

$V_2^H = 0$ for any nontrivial $H \subset G$

so Y is a free G-space

X and Y have G-CW structures where the one on Y is free.

Then W can be obtained from X by attaching free G-cells as follows

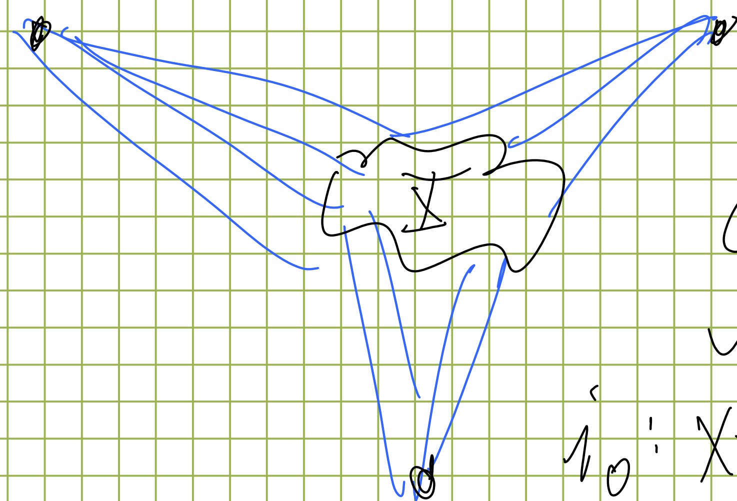
Let $Y^0 \subset Y^1 \subset Y^2 \subset \cdots Y^{d_2-1}$ be the skeleta of Y. Then

$$\begin{array}{ccccccc}
W^{d_1-1} & \hookrightarrow & W^{d_1} & \hookrightarrow & W^{d_1+1} & \hookrightarrow \cdots & W^{d_1+d_2-1} \\
\| & & \| & & \| & & \| \\
X & \xrightarrow{i_0} & X * Y^0 & \xhookrightarrow{i_1} & X * Y^1 & \hookrightarrow & X * Y
\end{array}$$

Pf: All but the first inclusion are functorial on the maps $Y^{j-1} \xhookrightarrow{i_j} Y^j$

Recall $X * Y = X \times I \times Y / \begin{array}{l}(x,0,y) \sim (x,0,y') \\ (x,1,y) \sim (x',1,y)\end{array}$

Picture for $Y_0 = C_3 = G$



$X * C_3$ = triple cone on X

G-actions on X and C_3 induce one on $X * C_3$

$i_0 : X \to X * C_3$ inclusion.

When $X = S^{d_1 - 1}$, we are attaching three d_1-cells to get $X * C_3$.

Similar thing happens at each stage and we have a G-CW structure on W

QED.

Back to C_4. Want to find $\underline{H}_* S^{m\sigma + n\lambda}$ $m, n \geq 0$

We have a G-CW structure on

$S^{m\sigma}$ and want to extend it to one on $S^{m\sigma + n\lambda}$. The Lemma applies because $n\lambda$ is a free rep of $C_4 = G$.

Let $G' \subset G$ be index 2 subgp

e.g. $m = 2 \quad n = 2$

$$\mathbb{Z} \xleftarrow{\nabla} \mathbb{Z}G/G' \xleftarrow{1-\gamma} \mathbb{Z}G/G' \xleftarrow{1+\gamma} \mathbb{Z}G \xleftarrow{1-\gamma} \mathbb{Z}G \xleftarrow{(1+\gamma)(1+\gamma^2)} \mathbb{Z}G \xleftarrow{1-\gamma} \mathbb{Z}G$$

0 1 2 3 4 5 6

$\underline{H}_*(S^{2\sigma+2\lambda})$ underlain by $H_* S^6$

$2\rho_4 = 2 + 2\sigma + 2\lambda$

$\mathbb{Z}G/G' = \mathbb{Z}\{1, \gamma\} \qquad (\mathbb{Z}G/G')^G = \mathbb{Z}\{1+\gamma\}$

$$(\mathbb{Z}G)^{G'} = \mathbb{Z}\{1+\gamma^2, \gamma+\gamma^3\} \qquad (\mathbb{Z}G)^{G} = \mathbb{Z}\{1+\gamma+\gamma^2+\gamma^3\}$$

$$\begin{tikzcd}
\mathbb{Z} \arrow[d, shift right, "1"'] & \mathbb{Z} \arrow[l, "2"'] \arrow[d, shift right, "1+\gamma"'] & \mathbb{Z} \arrow[l, "0"'] \arrow[d, shift right, "1+\gamma"'] & \mathbb{Z} \arrow[l, "4"'] \arrow[d, shift right, "1+\gamma"'] & \mathbb{Z} \arrow[l, "0"'] \arrow[d, shift right, "1+\gamma"'] & \mathbb{Z} \arrow[l, "4"'] \arrow[d, shift right] & \mathbb{Z} \arrow[l, "0"'] \arrow[d, shift right] \\
\mathbb{Z} \arrow[u, shift right, "2"'] \arrow[d, shift right, "1"'] & \mathbb{Z}G/G' \arrow[l, "\nabla"'] \arrow[u, shift right, "\nabla"'] \arrow[d, shift right, "1"'] & \mathbb{Z}G/G' \arrow[l, "1-\gamma"'] \arrow[u, shift right, "\nabla"'] \arrow[d, shift right, "1"'] & \mathbb{Z}G/G' \arrow[l, "2(1+\gamma)"'] \arrow[u, shift right, "\nabla"'] \arrow[d, shift right, "1+\gamma^2"'] & \mathbb{Z}G/G' \arrow[l, "1-\gamma"'] \arrow[u, shift right, "\nabla"'] \arrow[d, shift right, "1+\gamma^2"'] & \mathbb{Z}G/G' \arrow[l, "2(1+\gamma)"'] \arrow[u, shift right] \arrow[d, shift right] & \mathbb{Z}G/G' \arrow[l, "1-\gamma"'] \arrow[u, shift right] \arrow[d, shift right] \\
\mathbb{Z} \arrow[u, shift right, "2"'] & \mathbb{Z}G/G' \arrow[l, "\nabla"'] \arrow[u, shift right, "2"'] & \mathbb{Z}G/G' \arrow[l, "1-\gamma"'] \arrow[u, shift right, "2"'] & \mathbb{Z}G \arrow[l, "1+\gamma"'] \arrow[u, shift right, "\nabla"'] & \mathbb{Z}G \arrow[l, "1-\gamma"'] \arrow[u, shift right] & \mathbb{Z}G \arrow[l, "(1+\gamma)(1+\gamma^2)"'] \arrow[u, shift right] & \mathbb{Z}G \arrow[l, "1-\gamma"'] \arrow[u, shift right]
\end{tikzcd}$$

$$\begin{array}{ccccccc}
2 & 3 & 4 & 5 & 6 & 7 & 8
\end{array}$$

$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[d, shift right] & 0 \arrow[d, shift right] & \mathbb{Z}/4 \arrow[d, shift right, "1"'] & 0 \arrow[d, shift right] & \mathbb{Z}/4 \arrow[d, shift right, "1"'] & 0 \arrow[d, shift right] & \mathbb{Z} \arrow[d, shift right, "1"'] \\
0 \arrow[u, shift right] \arrow[d, shift right] & 0 \arrow[u, shift right] \arrow[d, shift right] & \mathbb{Z}/2 \arrow[u, shift right, "2"'] \arrow[d, shift right] & 0 \arrow[u, shift right] \arrow[d, shift right] & \mathbb{Z}/2 \arrow[u, shift right, "2"'] \arrow[d, shift right] & 0 \arrow[u, shift right] \arrow[d, shift right] & \mathbb{Z} \arrow[u, shift right, "2"'] \arrow[d, shift right, "1"'] \\
0 \arrow[u, shift right] & 0 \arrow[u, shift right] & 0 \arrow[u, shift right] & 0 \arrow[u, shift right] & 0 \arrow[u, shift right] & 0 \arrow[u, shift right] & \mathbb{Z} \arrow[u, shift right, "2"']
\end{tikzcd}$$