LARSON on $H\underline{M}$

Note Title 10/17/2012

How to construct an Eilenberg-Mac Lane spectrum $H\underline{M}$ for a Mackey functor $\underline{M}$.

An $RO(G)$ graded cohom theory is a cont. functor

$$E_G^* : G\text{-spectra} \longrightarrow RO(G)\text{-graded abelian gps}$$

satisfying the Eilenberg-Steenrod axioms

Thm (Brown) A functor k as above is representable by a G-spectrum K. $k(X) = [X, K]_G$ iff k satisfies wedge axiom and MVS.

<u>Cor</u> $\bar{E}^*_G$ as above is represented by a G-spectrum

<u>Cor</u> Any $\mathbb{Z}$-graded theory F on G-spectra is represented by a G-spectrum and it extends to an $RO(G)$-graded theory by $F^V(X) = F^0(\Sigma^{-V} X)$

<u>Thm</u> Given a Mackey functor $\underline{M}$ there is a spectrum $H\underline{M}$ with

$$\underline{\pi}_i H\underline{M} = \begin{cases} \underline{M} & i = 0 \\ 0 & i \neq 0 \end{cases} \quad \text{for } i \in \mathbb{Z}$$

Pf Will produce a $\mathbb{Z}$-graded theory:

X is a G-CW spectrum

$X^n = n$-skeleton

Define a chain cx of Mackey functors by

$\underline{C}_n(X) = \underline{\pi}_n(X^n / X^{n-1})$ with boundary operator d

Note X^n / X^{n-1} = wedge of n-sphere G-spectra

$$S^n_H = G/H_+ \wedge S^n$$

We dualize this chain cx

$C^n_G(X; \underline{M}) = \mathrm{Hom}(\underline{C}_n(X), \underline{M})$ with coboundary δ

This is a cochain cx of abelian gps.

Call its cohom $H^*_G(X, \underline{M})$

It is easy to check this satisfies the axioms required by Brown. We get a dimension axiom as follows

$$S^0_H = G/H_+ \wedge S^0 = G/H_+$$

$$\underline{C}_i(G/H_+) = 0 \text{ for } i \neq 0$$

$$\underline{C}_0(G/H_+) = \underline{\pi}_0(G/H_+)$$

$$\underline{\pi}_0(G/H_+)(G/K) = \pi_0\left((G/H)^K\right)$$

$$= \begin{cases} 0 & \text{if } K \not\subset H \\ \mathbb{Z}G/H & \text{if } K \subset H \end{cases}$$

$$C^i(G/H_+, \underline{M}) = \begin{cases} 0 & i \neq 0 \\ \mathrm{Hom}\left(\underline{C}_0(G/H_+), \underline{M}\right) & i = 0 \end{cases}$$

$$= \begin{cases} 0 & i \neq 0 \\ \underline{M}(G/H) & i = 0 \end{cases}$$

Hence we have $H^{*}_{G}(S^{0}_{M}, \underline{M}) = \underline{M}(G/H)$ (*)

This theory is representable by a G-spectrum $H\underline{M}$ with $\underline{\pi}_*$ concentrated in dim 0.

(*) $\Rightarrow$ $\underline{\pi}_i H\underline{M} = \begin{cases} \underline{M} & i = 0 \\ 0 & i \neq 0 \end{cases}$

since $\underline{\pi}_0(H\underline{M})(G/H) = [G/H_+, H\underline{M}]^{0}_{G}$

$$= H^{0}_{G}(G/H_+, \underline{M}) = M(G/H)$$

by previous computation QED