Note Title 10/19/2012

Fixed points

For a G-space X,

$$X^G = \{x \in X : g(x) = x \ \forall g \in G\}$$

$$= \mathrm{Map}^G(pt, X) = \text{ordinary fixed point set}$$

Suppose we replace pt by a contractible free G-space, e.g. $EG = G * G * G * \cdots$

$$X^{hG} = \mathrm{Map}^G(EG, X) = \text{homotopy fixed point set}$$

The equivariant map $EG \xrightarrow{\varphi} pt$

inducing $X^G \longrightarrow X^{hG}$

It may or may not be an equivalence

φ is not an G-equiv

Recall a G-map $X \xrightarrow{f} Y$ is a G-equiv

iff $X^H \longrightarrow Y^H$ is an equiv for all $H < G$

φ does not do this because

$$EG^H = \begin{cases} EG \simeq * & \text{if } H = e \\ \emptyset & \text{if } H \neq e \end{cases}$$

$$pt^H = pt \quad \text{for any } H$$

Example G acts trivially on X

$X^{hG} = \mathrm{Map}^G(EG, X) = \mathrm{Map}(EG/G, X)$ orbit space

$= \mathrm{Map}(BG, X)$

$X^G = X$

Sullivan Conjecture (proved by Miller in 1986)

For G finite and X is a finite CX then $\mathrm{Map}(BG, X) \simeq X$. All maps $BG \to X$ are null homotopic.

$G = C_n$, $X = K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$

$\pi_0(\mathrm{Map}(BG, \mathbb{C}P^\infty)) = [BG, K(\mathbb{Z}, 2)]$

$= H^2(BG; \mathbb{Z}) = \mathbb{Z}/n$

$(\mathbb{C}P^\infty)^{hG}$ has n components

$(\mathbb{C}P^\infty)^{G} = \mathbb{C}P^\infty$ has 1 component.

There is an analogous definition for spectra

$X^{hG} = \mathrm{Map}^G(EG_+, X)$ for a spectrum X.

Thm 1 There is a homotopy fixed point SS

$$\pi_* X^{hG} \Longleftarrow H^*(G; \pi_* X)$$

($\pi_* X$ is a $\mathbb{Z}G$-module)

Thm 2 Suppose $X \xrightarrow{f} Y$ is G-map underlain by an equivalence.

Then $X^{hG} \longrightarrow Y^{hG}$ is an equivalence.

Partial proof. Let F be the fiber of f.
It is a contractible G-space and
$\mathrm{Map}(EG, F)$ is the fiber of the map

$$\mathrm{Map}(EG, X) \xrightarrow{f_*} \mathrm{Map}(EG, Y)$$

$\mathrm{Map}^G(EG, F)$ is the fiber of $X^{hG} \longrightarrow Y^{hG}$

$$\begin{gathered} \mathrm{Map}^G(EG,F) \\ \parallel \\ F^{hG} \end{gathered}$$

This reduces the thm to:
If F is a contractible G-space
then $F^{hG} \simeq *$. (end of partial proof)

Geometric fixed pts

Recall ordinary fixed pts behave badly in spectra

$$(X \wedge Y)^G \neq X^G \wedge Y^G$$

$G = C_4$ $\quad (S^{\rho_4} \wedge H\underline{\mathbb{Z}})^G \neq (S^{\rho_4})^G \wedge (H\underline{\mathbb{Z}})^G = S^1 \wedge H\mathbb{Z}$

$$(\Sigma^\infty_G X)^G \neq \Sigma^\infty (X^G)$$

e.g. X has trivial G-action, so $\Sigma^\infty(X^G) = \Sigma^\infty X$

for $E = \Sigma^\infty_G X$, $E_V = S^V \wedge X = \Sigma^V X$

G acts nontrivially on E, which has a different fixed pt set from $\Sigma^\infty X$.

The geometric fixed pt spectrum $\Phi^H(X)$ for each $H < G$ satisfies

1) $\Phi^H(X \wedge Y) = \Phi^H(X) \wedge \Phi^H(Y)$

2) $\Phi^H(\Sigma^\infty_G X) = \Sigma^\infty(X^H)$

3) G map $X \xrightarrow{f} Y$ is a G-equiv iff $\Phi^H f$ is an equivalence for each H.

Def A family $\mathcal{F}$ of subgps of G is a collection of subgps closed under inclusion and conjugation, i.e.

1) if $H \in \mathcal{F}$ and $K \subset H$ then $K \in \mathcal{F}$

2) if $H \in \mathcal{F}$ then $gHg^{-1} \in \mathcal{F}$.

Examples 1) $\mathcal{F} = \{e\}$, $\mathcal{F}$ = all subgps

2) $\mathcal{P}$ = all proper subgps

3) $\mathcal{F}_p$ = all p-subgps.

Suppose we have a G-space $E\mathcal{F}$

with $E\mathcal{F}^H = \begin{cases} \emptyset & \text{for } H \notin \mathcal{F} \\ \text{contractible} & \text{for } H \in \mathcal{F} \end{cases}$

example $G = C_{p^n}$ $\mathcal{F} = \mathcal{P}$ $G' = C_{p^{n-1}}$

$E\mathcal{P}^H = \begin{cases} \emptyset & \text{if } H = G \\ \text{contr} & \text{if } H \neq G \end{cases}$

$E(G/G') = E\mathcal{P}$

There is a way to construct $E\mathcal{F}$ as an infinite join of ~~spaces~~ spaces of the form G/H for $H \in \mathcal{F}$.