Note Title 10/29/2012

Recall X^{hG} = homotopy fixed point set

Geometric fixed points $\Phi^G X$ (not yet defined) have more convenient properties than ordinary fixed points.

For a family (a collection closed under conjugations + inclusions) $\mathcal{F}$ of subgps of G there is a G-space $E\mathcal{F}$ with

$$E\mathcal{F}^H = \begin{cases} \phi & \text{for } H \notin \mathcal{F} \\ \text{contractible} & \text{for } H \in \mathcal{F} \end{cases}$$

e.g. $\mathcal{P}$ = family of proper subgps of G.

For $G = C_{p^n}$, let $G' \subset G$ be the index p subgp. Then $EP = E(G/G')$.

G acts on the contractible free G/G' then the quotient. Hence each

REVIEW

proper subgp of G acts trivially and hence has a contractible fixed pt set, while G acts with fixed points.

Consider the cofiber sequence of G-spaces

$$EP \longrightarrow pt \longrightarrow \tilde{E}P$$

Analogously for spectra we have

$E\mathcal{P}_+ \longrightarrow S^0 \longrightarrow \tilde{E}\mathcal{P}$ = suspension spectrum of the space $\tilde{E}\mathcal{P}$

ISOTROPY SEPARATION SEQUENCE

Introduced by Carlsson in 1982

$$E\mathcal{P}_+^H = \begin{cases} * & \text{if } H = G \\ S^0 & \text{if } H \neq G \end{cases}$$

$(S^0)^H = S^0$ for all $H \leq G$

$$\tilde{E}\mathcal{P}^H = \begin{cases} S^0 & \text{for } H = G \\ * & \text{for } H \neq G \end{cases}$$

For $G = C_{2^n}$, $G/G' \cong C_2$

$E\mathcal{P} = EG/G' = S^\infty$ with antipodal G-action

$\tilde{E}\mathcal{P} \simeq \Sigma S^\infty$ (fixed pt set is S^0)

$= \varinjlim (S^0 \to S^6 \to S^{2\sigma} \to S^{3\sigma} \to \cdots)$

where σ = sign rep

Def For a G-spectrum X,

$$\Phi^G X = (\tilde{E}\mathcal{P} \wedge X)^G$$

Thm Properties of Φ^G (functor from G-spectra to ordinary spectra)

i) The Φ^G preserves homotopy direct limits

ii) Φ^G preserves weak equivalences

iii) For a G-spectrum X and a G-space T,

$$\Phi^G(X \wedge T) = \Phi^G(X) \wedge T^G$$

e.g. $\Phi(\Sigma^\infty T) = \Sigma^\infty(T^G)$

iv) $\Phi^G(X \wedge Y) = \Phi^G(X) \wedge \Phi^G(Y)$

v) $\Phi^G(S^{-V}) = S^{-V_0}$ where $V_0 = V^G$ for a representation V.

Thm For $G = C_{2^n}$, let σ be the sign rep. Recall $a_\sigma : S^0 \to S^\sigma$ is an element in $\pi_0^G S^\sigma = \pi_{-\sigma}^G S^0 = \underline{\pi}_{-\sigma} S^0(G/G)$

Then the $\mathbb{Z}$-graded portion of
$a_\sigma^{-1} \underline{\pi}_\bigstar X(G/G)$ (an $RO(G)$-graded abelian gp)
is $\pi_* \Phi^G \underline{X}$

Proof Recall $\tilde{E}P$ is the direct limit of

$$S^0 \xrightarrow{a} S^\sigma \xrightarrow{a \wedge S^\sigma} S^{2\sigma} \xrightarrow{a \wedge S^{2\sigma}} S^{3\sigma} \longrightarrow \cdots$$

where $a = a_\sigma$

Inverting a in $\underline{\pi}_* X(G/G)$ is the the same as smashing with

$E\tilde{P}$ and getting $\pi_*(X \wedge E\tilde{P})(G/G)$

$$= \pi_*(X \wedge E\tilde{P})^G$$

$$= \pi_*(\Phi^G X).$$

Long winded example: real K-theory as defined by Atiyah in "K-theory and reality"

Recall $U(n)$ = gp of complex unitary $(n \times n)$-matrices unitary means $M^{-1} = \overline{M}^t$

$U(n) \hookrightarrow GL_n(\mathbb{C})$ is a htpy equivalence

$\shortparallel$ compact Lie gp of dim n^2-1 $\qquad$ $\shortparallel$ non-compact Lie gp of dim $2n^2$

The maps $U(n) \to U(n+1) \to U(n+2) \to \cdots$

induce $BU(n) \to BU(n+1) \to \cdots$,

with direct limit BU.

$$\pi_i(BU) = \begin{cases} \mathbb{Z} & \text{for } i \text{ even} \\ 0 & \text{for } i \text{ odd} \end{cases}$$

(Bott Periodicity thm)

$\Omega^2 BU = \mathbb{Z} \times BU$ DEEP THEOREM!

For each n there is a $(n-1)$-conn. space X_n with $\Omega^n X_n \simeq \mathbb{Z} \times BU$, i.e.
$\mathbb{Z} \times BU$ is an infinite loop space
These form a spectrum bu or k,
connective complex K-theory.
This can be made into a C_2
equivariant spectrum bu
There is also a periodic version
i.e. a C_2-spectrum
K with

$$\pi_i^u K = \left\{\begin{array}{ll} \mathbb{Z} & \text{for } i \text{ even} \\ 0 & \text{for } i \text{ odd} \end{array}\right\} \text{ for all } i \in \mathbb{Z}$$

Note $U(n)^{C_2} = O(n)$

$$BU^{C_2} = BO$$

where O = stable orthogonal gp.

Bott proved

$$\pi_i BO = \begin{cases} \mathbb{Z} & \text{for } i = 0 \bmod 4 \\ \mathbb{Z}/2 & \text{for } i = 1,2 \bmod 8 \\ 0 & \text{else} \end{cases}$$

$\mathbb{Z}/2 \quad \mathbb{Z}/2 \quad 0 \quad \mathbb{Z} \quad 0 \quad 0 \quad 0 \quad \mathbb{Z}$

and $\Omega^8 BO = \mathbb{Z} \times BO$

There are spectra bo and KO with similar properties the bu and K

For $G = C_2$, $bu^G = bo$ and $K^G = KO$.