MATH 550

Note Title 10/3/2012

Volunteers needed for Thm 2.14 and the construction of the Eilenberg-Mac Lane spectrum $H\underline{M}$.

Back to "homology of a point"

i.e $\underline{H}_\star S^0 = \underline{\pi}_\star H\underline{\mathbb{Z}}$

$= RO(G)$ graded gp

Classically $\underset{\text{"}\pi_i H\mathbb{Z}\text{"}}{H_i S^0} = \begin{cases} \mathbb{Z} & \text{for } i=0 \\ 0 & \text{for } i \neq 0 \end{cases}$

Dimension axiom.

We know

$$\underline{\pi}_i\, H\underline{\mathbb{Z}} = \begin{cases} \underline{\mathbb{Z}} & \text{for } i=0 \\ 0 & \text{for } i\neq 0 \end{cases} \Big\}\ \text{for } i\in\mathbb{Z}$$

Recall (from 9/10/12) the calculations of $\underline{H}_* S^{n\sigma}$ for $n\geq 0$ where $G=C_2$ and σ = sign rep.

$$\underline{H}_i S^{n\sigma} = \underline{\pi}_i\, S^{n\sigma}\wedge H\underline{\mathbb{Z}} = \underline{\pi}_{i-n\sigma}\, H\underline{\mathbb{Z}}$$

We want to know

$\underline{\pi}_\star H\underline{\mathbb{Z}}$ i.e. $\underline{\pi}_{i+j\sigma} H\underline{\mathbb{Z}}$ for all $i,j\in\mathbb{Z}$

we have it for $j\leq 0$ but not $j>0$.

$S^{n\sigma}$ is a G-CW cx leading to a reduced cellular chain cx of

$\mathbb{Z}G$ - modules ($\mathbb{Z}G = \mathbb{Z}[G]$)

$$\begin{tikzcd}
\mathbb{Z} \arrow[d, bend right, "1"'] & \mathbb{Z} \arrow[l, "2"'] \arrow[d, bend right, "1+\gamma"'] & \mathbb{Z} \arrow[l, "0"'] \arrow[d, bend right, "1+\gamma"'] & \mathbb{Z} \arrow[l, "2"'] \arrow[d, bend right, "1+\gamma"'] & \mathbb{Z} \arrow[l, "0"'] \arrow[d, bend right] \\
\mathbb{Z} \arrow[u, bend right, "2"'] & \mathbb{Z}G \arrow[l, "\nabla"'] \arrow[u, bend right, "\nabla"'] & \mathbb{Z}G \arrow[l, "1-\gamma"'] \arrow[u, bend right, "\nabla"'] & \mathbb{Z}G \arrow[l, "1+\gamma"'] \arrow[u, bend right, "\nabla"'] & \mathbb{Z}G \arrow[l, "1-\gamma"'] \arrow[u, bend right]
\end{tikzcd}$$

0 1 2 3 $n=3$

In homology we get

B $\qquad \overline{\underline{\mathbb{Z}}} \qquad \underline{\mathbb{Z}}$

$n=3$
$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[d, bend right] & 0 \arrow[d, bend right] & \mathbb{Z}/2 \arrow[d, bend right] & 0 \arrow[d, bend right] \\
0 \arrow[u, bend right] & 0 \arrow[u, bend right] & 0 \arrow[u, bend right] & \mathbb{Z}_- \arrow[u, bend right]
\end{tikzcd}$$

$n=4$
$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[d, bend right] & 0 \arrow[d, bend right] & \mathbb{Z}/2 \arrow[d, bend right] & 0 & \mathbb{Z} \arrow[d, bend right, "1"'] \\
0 \arrow[u, bend right] & 0 \arrow[u, bend right] & 0 \arrow[u, bend right] & 0 & \mathbb{Z} \arrow[u, bend right, "2"']
\end{tikzcd}$$

We need names for these 3 nontrivial

Mackey functions indicated in red.

$$\underline{H}_i\, S^{n\sigma} = \begin{cases} \underline{B} & \text{for } i \text{ even and } 0 \le i < n \\ \underline{\underline{Z}}_- & \text{for } i = n \text{ and } n \text{ odd} \\ \underline{Z} & \text{for } i = n,\ n \text{ even.} \end{cases}$$

for $n \geq 0$.

$\underline{\pi}_\star\, H\underline{Z}$ is an $RO(G)$-graded ring

$\quad \| $

$\underline{H}_\star\, S^0$

This is called an $RO(G)$-graded Green functor.

i.e. $\alpha \in \underline{\pi}_V\, H\underline{Z}(G/H)$

$\beta \in \underline{\pi}_W\, H\underline{Z}(G/H)$

then $\alpha\beta \in \underline{\pi}_{V+W}\, H\underline{Z}(G/H)$

If $\underline{M}$ is a Green functor, i.e. $\underline{M}(G/H)$ is a ring for each H

Then for $H \subset K \subset G$

$$\begin{tikzcd} \underline{M}(G/H) \arrow[rr, bend right, "\mathrm{Tr}_H^K"'] & & \underline{M}(G/K) \arrow[ll, bend right, "\mathrm{Res}_H^K"'] \end{tikzcd}$$

Res_H^K (but not Tr_H^K) is a ring hom.

Some handy elements in $\pi_\star H\underline{\mathbb{Z}}$

1) Let V be a rep with $V^G = 0$.

Then $(S^V)^G = S^0$ so we have

an equivariant $S^0 \xrightarrow{a_V} S^V$

$$a_V \in \underline{\pi}_0 S^V(G/G) = \underline{\pi}_{-V} S^0(G/G)$$

It has a Hurewicz image in

$$\underline{H}_{-V} S^0(G/G) = \underline{\pi}_{-V} H\underline{\mathbb{Z}}(G/G)$$

e.g. $G = C_2$, $V = \sigma$ $\quad a_\sigma \in \pi_{-\sigma} H\underline{\mathbb{Z}}(G/G)$

2) Let V be an oriented rep
i.e. determinant of each matrix is 1
e.g. $V = 2n\sigma$. Let $|V| = \dim V$. Then

$$\underline{H}_{|V|} S^V = \underline{H}_{|V|-V} S^0 = \underline{\pi}_{|V|-V} H\underline{\mathbb{Z}} = \underline{\mathbb{Z}}$$

where $\underline{\mathbb{Z}}(G/H) = \mathbb{Z}$ for all H

and each restriction map is iso.

Let u_V be a generator of

$$\pi_{|V|-V} H\underline{\mathbb{Z}}(G/G)$$

Relaxing the two conditions on V:

1) If $V^G = \mathbb{R}^n$, then we get a

map $S^n \xrightarrow{a_V} S^V$ in

$$\underline{\pi}_n S^V(G/G) = \underline{\pi}_{n-V} S^0(G/G) \to \underline{H}_{n-V} S^0(G/G)$$

$V = n + W$ where $W^G = 0$

In $RO(G)$, $n - V = n - (n+W) = -W$

$a_V = a_W$.

2) If V is not oriented, e.g. $V = \sigma$

We have a hom $G \xrightarrow{\det} \{\pm 1\}$

Its kernel H has index 2

$\underline{H}_{|V|} S^V = \underline{\underline{Z}}$ defined by

$$\underline{\underline{Z}}(G/K) = \begin{cases} 0 & \text{if } K \not\subseteq H \\ Z_- & \text{if } K \subseteq H \end{cases}$$

where γ acts nontrivially on Z_-

for $\gamma \notin H$.