

MATH 550

Note Title

10/3/2012

Volunteers needed for Thm 2.14
and the construction of the
Eilenberg-Mac Lane spectrum HM.

Back to "homology of a point"

$$\text{i.e. } H_* S^0 = \underline{\mathbb{Z}} \oplus \underline{\mathbb{Z}}$$

$$= RO(G) \text{ graded gp}$$

$$\text{Classically } H_i S^0 = \begin{cases} \mathbb{Z} & \text{for } i=0 \\ 0 & \text{for } i \neq 0 \end{cases}$$

$\pi_i \mathbb{Z}$

Dimension axiom.

We know

$$\underline{\Pi}_i \underline{H}\underline{Z} = \begin{cases} \cong & \text{for } i=0 \\ 0 & \text{for } i \neq 0 \end{cases} \text{ for } i \in \underline{Z}$$

Recall (from 9/10/12) the calculation of $\underline{H}_* S^{n\mathbb{G}}$ for $n \geq 0$ where $G_1 = C_2$ and $\sigma = \text{sign rep.}$

$$\underline{H}_i S^{n\mathbb{G}} = \underline{\Pi}_i S^{n\mathbb{G}} \wedge \underline{H}\underline{Z} = \underline{\Pi}_{i-n\mathbb{G}} \underline{H}\underline{Z}$$

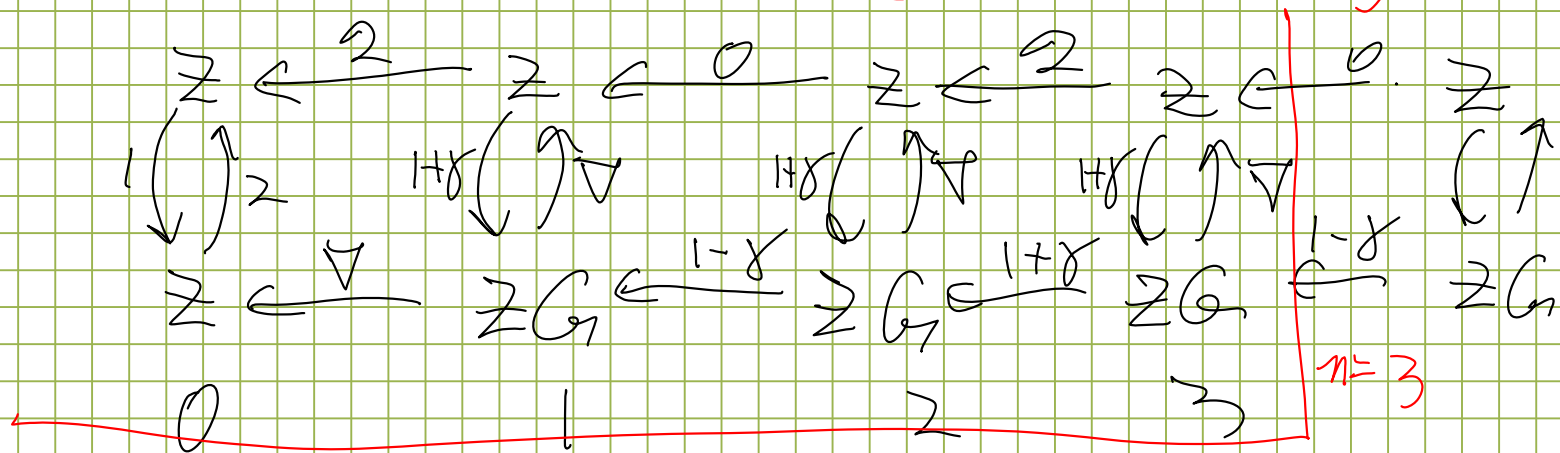
We want to know

$$\underline{\Pi}_i \star \underline{H}\underline{Z} \text{ i.e. } \underline{\Pi}_{i+j\mathbb{G}} \underline{H}\underline{Z} \text{ for all } i, j \in \underline{Z}$$

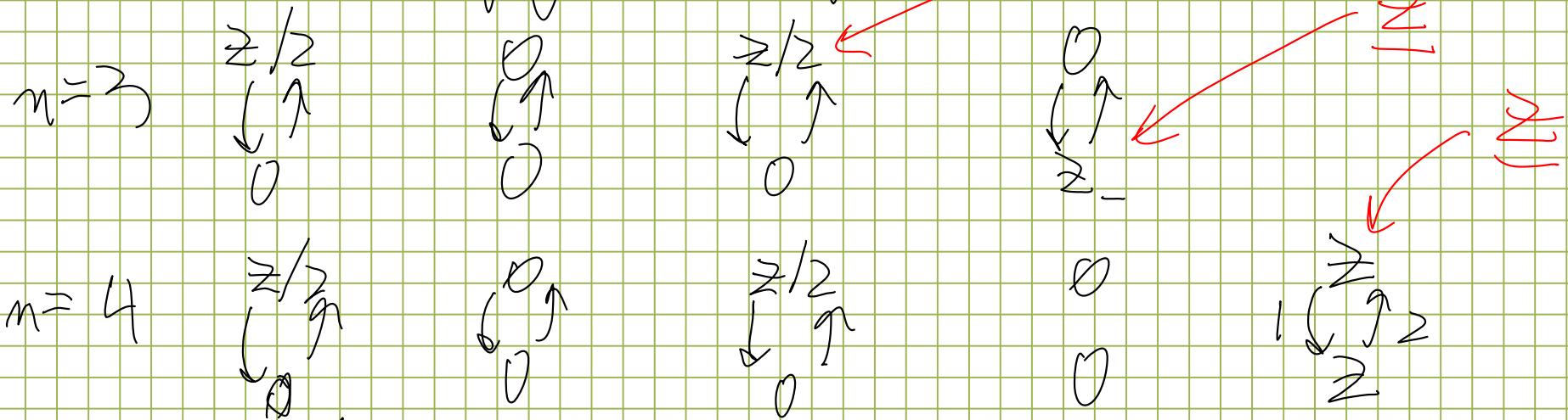
we have it for $j \leq 0$ but not $j > 0$.

$S^{n\mathbb{G}}$ is a G_1 -CW cx leading to a reduced cellular chain cx of

$\mathbb{Z}G$ -modules $(\mathbb{Z}G = \mathbb{Z}[G])$



In homology we get



We need names for these 3 nontrivial

Mackey functors indicated in *red*.

$$\underline{H}_i S^{nG} = \begin{cases} B & \text{for } i \text{ even and } 0 \leq i < n \\ \underline{\mathbb{Z}} & \text{for } i = n \text{ and } n \text{ odd} \\ \underline{\mathbb{Z}} & \text{for } i = n, n \text{ even.} \end{cases}$$

for $n \geq 0$.

$\prod_i \underline{H}_i S^{nG}$ is an $RO(G)$ -graded ring

$$\prod_i \underline{H}_i S^0$$

i.e. $\alpha \in \prod_v \underline{H}_v (G/H)$

$$\beta \in \prod_w \underline{H}_w (G/H)$$

then $\alpha \beta \in \prod_{v+w} \underline{H}_{v+w} (G/H)$

This is called
an $RO(G)$ -graded
Green functor.

If \underline{M} is a Green functor, i.e.

$\underline{M}(G/H)$ is a ring for each H

Then for $H \subset K \subset G$

$$\begin{array}{ccc} \underline{M}(G/H) & \xleftarrow{\text{Res}_H^K} & \underline{M}(G/K) \\ & \xrightarrow{\text{Tr}_H^K} & \end{array}$$

Res_H^K (but not Tr_H^K) is a ring hom.

Some handy elements in $\underline{\mathbb{Z}} \cong \underline{\mathbb{Z}} \cong \underline{\mathbb{Z}}$

1) Let V be a rep with $V^G = 0$.

Then $(S^V)^G = S^0$ so we have

an epimorphism $S^0 \xrightarrow{a_V} S^V$

$$a_V \in \underline{\Pi}_0 S^V(G_1/G_1) = \underline{\Pi}_{-V} S^0(G_1/G_1)$$

It has a Hurewicz image in

$$\underline{H}_{-V} S^0(G_1/G_1) = \underline{\Pi}_{-V} \underline{H}\mathbb{Z}(G_1/G_1)$$

e.g. $G_1 = G_2, V = 6 \quad a_6 \in \underline{\Pi}_{-6} \underline{H}\mathbb{Z}(G_1/G_1)$

2) Let V be an oriented rep
i.e. determinant of each matrix is 1
e.g. $V = \mathbb{Z}n\sigma$. Let $|V| = \dim V$. Then

$$\underline{H}_{|V|} S^V = \underline{H}_{|V|-V} S^0 = \underline{\Pi}_{|V|-V} \underline{H}\mathbb{Z} = \underline{\mathbb{Z}}$$

where $\mathbb{Z}(G/H) = \mathbb{Z}$ for all H
 and each restriction map is iso.
 Let μ_V be a generator of

$$\prod_{|H|=V} H\mathbb{Z}(G/G)$$

Relating the two conditions on V :

1) If $V^{G_1} = \mathbb{R}^n$, then we get a
 map $S^n \xrightarrow{a_V} S^V$ in

$$\prod_{|H|=n} S^V(G_1/G_1) = \prod_{|H|=n-V} S^0(G_1/G_1) \rightarrow \prod_{|H|=n-V} S^0(G_1/G_1)$$

$V = n + W$ where $W^{G_1} = 0$

$$\dim \operatorname{RO}(G), \quad n - V = n - (n + W) = -W$$

$$a_V = a_W.$$

2) If V is not oriented, s.g. $V = 0$

We have a hom $G \xrightarrow{\det} \{\pm 1\}$

Its kernel H has index 2

$\underline{\mathbb{Z}}_{|V|} \hookrightarrow \underline{\mathbb{Z}} = \underline{\mathbb{Z}}$ defined by

$$\underline{\mathbb{Z}}(G/K) = \begin{cases} 0 & \text{if } K \not\subseteq H \\ \underline{\mathbb{Z}}_- & \text{if } K \subseteq H \end{cases}$$

where γ acts nontrivially on $\underline{\mathbb{Z}}_-$
for $\gamma \notin H$.

