MATH 550

Note Title 10/5/2012

Assume G is a cyclic 2-gp

Recall we have

$$a_V \in \underline{\pi}_{-V} H\underline{\mathbb{Z}}(G/G) \quad \text{for } V^G = 0$$

$$u_V \in \underline{\pi}_{|V|-V} H\underline{\mathbb{Z}}(G/G_{\det V})$$

where G_V = isotropy gp of V

= largest subgp of G acting trivially on V

$\det(V)$ = determinant rep associated with V.

if V is oriented then $\det(V) = 1$

$G_{\det V} = G$, otherwise

$G_{\det V}$ has index 2.

Also we have $e_V \in \underline{\pi}_{V-|V|} H\mathbb{Z}(G/G_V)$ given by the underlying equiv

$$S^V \longrightarrow S^{|V|} \qquad |V| = \dim V$$

<u>Lemma 9</u> (in a HHR preprint)

i) $a_{V+W} = a_V a_W$ and $u_V u_W = u_{V+W}$

ii) $|G/G_V|\, a_V = 0$

iii) For oriented V, $\mathrm{Tr}_{G_V}^{G}(e_V)$ has infinite order, as does $\mathrm{Tr}_{G_{V+G}}^{G'}(e_{V+G})$

where G' is the subgp of index 2, which is $G_{\det\sigma} = G_\sigma$

$Tr_{G_{V+\sigma}}^{G}(e_{V+\sigma})$ has order 2 if $|V|>0$

$Tr_{G_\sigma}^{G}(e_\sigma) = 0$.

iv) For oriented V, $u_V \, Tr_{G_V}^{G}(e_V) = [G/G_V]$ in $\underline{\pi}_0 H\underline{\mathbb{Z}}(G/G) = \mathbb{Z}$

3 more similar statements.

Back to $G = C_2$. We have calculated

$\underline{\pi}_{i+j\sigma} H\underline{\mathbb{Z}}$ for $j \leq 0$, $i \in \mathbb{Z}$.

$$u_{2\sigma} \in \underline{\pi}_{2-2\sigma} H\underline{\mathbb{Z}}(G/G)$$

$$u_{\sigma} \in \underline{\pi}_{1-\sigma} H\underline{\mathbb{Z}}(G/e)$$

with $u_{\sigma}^2 = \mathrm{Res}^G_e \, u_{2\sigma}$

$$a_{\sigma} \in \underline{\pi}_{-\sigma} H\underline{\mathbb{Z}}(G/G)$$

Claim $u_{2\sigma}^i a_{\sigma}^j = u_{2i\sigma} a_{j\sigma}$ for $i, j \geq 0$

$$\in \underline{\pi}_{2i-(2i+j)\sigma} H\underline{\mathbb{Z}}(G/G)$$

$$= \begin{cases} \mathbb{Z} & \text{if } j = 0 \\ \mathbb{Z}/2 & \text{if } j > 0 \end{cases}$$

In each case this element generates the gp.

$$u_\sigma^{i} \in \pi_{i-i\sigma} H\underline{\mathbb{Z}}(G/e) = \mathbb{Z}$$

is the generator.

To get at $\pi_{i+j\sigma} H\underline{\mathbb{Z}}$ for $j>0$ we need to find $\underline{H}_*(S^{-j\sigma})$ for $j>0$

$S^{-j\sigma}$ is the Spanier-Whitehead dual of $S^{j\sigma}$, i.e. $S^{-j\sigma} \wedge S^{j\sigma} = S^0$

A cellular chain cx for $S^{-j\sigma}$ can be obtained by dualizing the one for $S^{j\sigma}$. ("dualizing" to be explained)

$C_* S^{j\sigma}$ has the form

$$\begin{array}{ccccccccc} 0 & & 1 & & 2 & & 3 & & 4 \\ \mathbb{Z} & \xleftarrow{\nabla} & \mathbb{Z}G & \xleftarrow{1-\gamma} & \mathbb{Z}G & \xleftarrow{1+\gamma} & \mathbb{Z}G & \xleftarrow{1-\gamma} & \mathbb{Z}G \leftarrow \cdots \end{array}$$

apply $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ to this and get a cochain cx

$$\begin{array}{ccccccccc} \mathbb{Z} & \xrightarrow[1+\gamma]{\Delta} & \mathbb{Z}G & \xrightarrow{1-\gamma} & \mathbb{Z}G & \xrightarrow{1+\gamma} & \mathbb{Z}G & \xrightarrow{1-\gamma} & \mathbb{Z}G \to \cdots \\ 0 & & -1 & & -2 & & -3 & & -4 \end{array}$$

This is a cellular chain cx for $S^{-j\sigma}$

apply the fixed point Mackey functor and

$0 \quad -1 \quad -2 \quad -3$

$j=2$ $\qquad$ $j=3$

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "1"] \arrow[d, bend right, "1"'] & \mathbb{Z} \arrow[r, "0"] \arrow[d, bend right, "\Delta"'] & \mathbb{Z} \arrow[r, "2"] \arrow[d, bend right, "\Delta"'] & \mathbb{Z} \arrow[r, "0"] \arrow[d, bend right, "\Delta"'] & \mathbb{Z} \arrow[r, "2"] \arrow[d, bend right, "\Delta"'] & \cdots \\
\mathbb{Z} \arrow[r, "\Delta", "1+\gamma"'] \arrow[u, bend right, "2"'] & \mathbb{Z}G \arrow[r, "1-\gamma"] \arrow[u, bend right, "\nabla"'] & \mathbb{Z}G \arrow[r, "1+\gamma"] \arrow[u, bend right, "\nabla"'] & \mathbb{Z}G \arrow[r, "1-\gamma"] \arrow[u, bend right, "\nabla"'] & \mathbb{Z}G \arrow[r] \arrow[u, bend right, "\nabla"'] & \cdots
\end{tikzcd}$$

$\underline{H}_*$

$$\begin{tikzcd}
\mathbb{Z} \arrow[d, bend right] & \mathbb{Z} \arrow[l, "2"'] \arrow[d, bend right] & \mathbb{Z} \arrow[l, "0"'] \arrow[d, bend right] & \mathbb{Z} \arrow[l, "2"'] \arrow[d, bend right] & \cdots \arrow[l] \\
\mathbb{Z} \arrow[u, bend right, "2"'] & \mathbb{Z}G \arrow[l, "\Delta"'] \arrow[u, bend right] & \mathbb{Z}G \arrow[l, "1-\gamma"'] \arrow[u, bend right] & \mathbb{Z}G \arrow[l] \arrow[u, bend right] &
\end{tikzcd}$$

$$\begin{array}{cccc}
\begin{tikzcd} 0 \arrow[d, bend right] \\ 0 \arrow[u, bend right] \end{tikzcd} & \begin{tikzcd} 0 \\ 0 \end{tikzcd} & \begin{tikzcd} 0 \\ 0 \end{tikzcd} & \begin{tikzcd} \mathbb{Z}/2 \arrow[d, bend right, "0"'] \\ \mathbb{Z}_- \arrow[u, bend right, "1"'] \end{tikzcd} \\
0 & -1 & -2 & -3
\end{array}$$

$\underline{H}_* S^{-3\sigma}$

New Mackey functor

$$\begin{array}{ccccc}
\begin{tikzcd} 0 \\ 0 \end{tikzcd} & \begin{tikzcd} 0 \\ 0 \end{tikzcd} & \begin{tikzcd} 0 \\ 0 \end{tikzcd} & \begin{tikzcd} \mathbb{Z}/2 \\ 0 \end{tikzcd} & \begin{tikzcd} \mathbb{Z} \arrow[d, bend right, "2"'] \\ \mathbb{Z} \arrow[u, bend right, "1"'] \end{tikzcd} \\
0 & -1 & -2 & -3 & -4
\end{array}$$

$\underline{H}_* S^{-4\sigma}$

$\underline{M}$

$$\begin{tikzcd}
M(G/G) \arrow[d, bend right, "\mathrm{Res}^{Z}_{1}"'] \\
M(G/e) \arrow[u, bend right, "\mathrm{Tr}^{Z}_{1}"']
\end{tikzcd}$$

$= \mathbb{Z}G$-module