Note Title 11/9/2012

Alternate interpretation of the FGL for complex cobordism

Let M be a complex mfd with a complex line bundle λ. λ is classified by a map

$$M \xrightarrow{f} BU(1) = \mathbb{C}P^\infty = K(\mathbb{Z}, 2)$$

If M has dimension $2n$ then the map f factors uniquely thru $\mathbb{C}P^n$.

$$\begin{array}{ccc} M & \xrightarrow{f} & \mathbb{C}P^n \\ \uparrow & & \uparrow \\ f^{-1}(\mathbb{C}P^{n-k}) & \longrightarrow & \mathbb{C}P^{n-k} \end{array} \qquad k=1,2,\dots n$$

A ~~thm~~ says f is homotopic to a map in which $f^{-1}(\mathbb{C}P^{n-k})$ is a cx submanifold of real codim $2k$. We get a sequence of cx submfds $M = N^0 \supset N^1 \supset N^2 \supset N^3 \dots$ and the normal of N^{k+1} in N^k

is the restriction of λ to ν^{k+1}!

$$M \xrightarrow{f} \mathbb{C}P^\infty = BU(1) = MU(1)$$

$MU(1)$ = Thom space for the taut. line bundle / $\mathbb{C}P^\infty$

We have maps $MU(1) \to \Omega^2 MU(2) \to \Omega^4 MU(3) \to \cdots$

Hence we get a map to

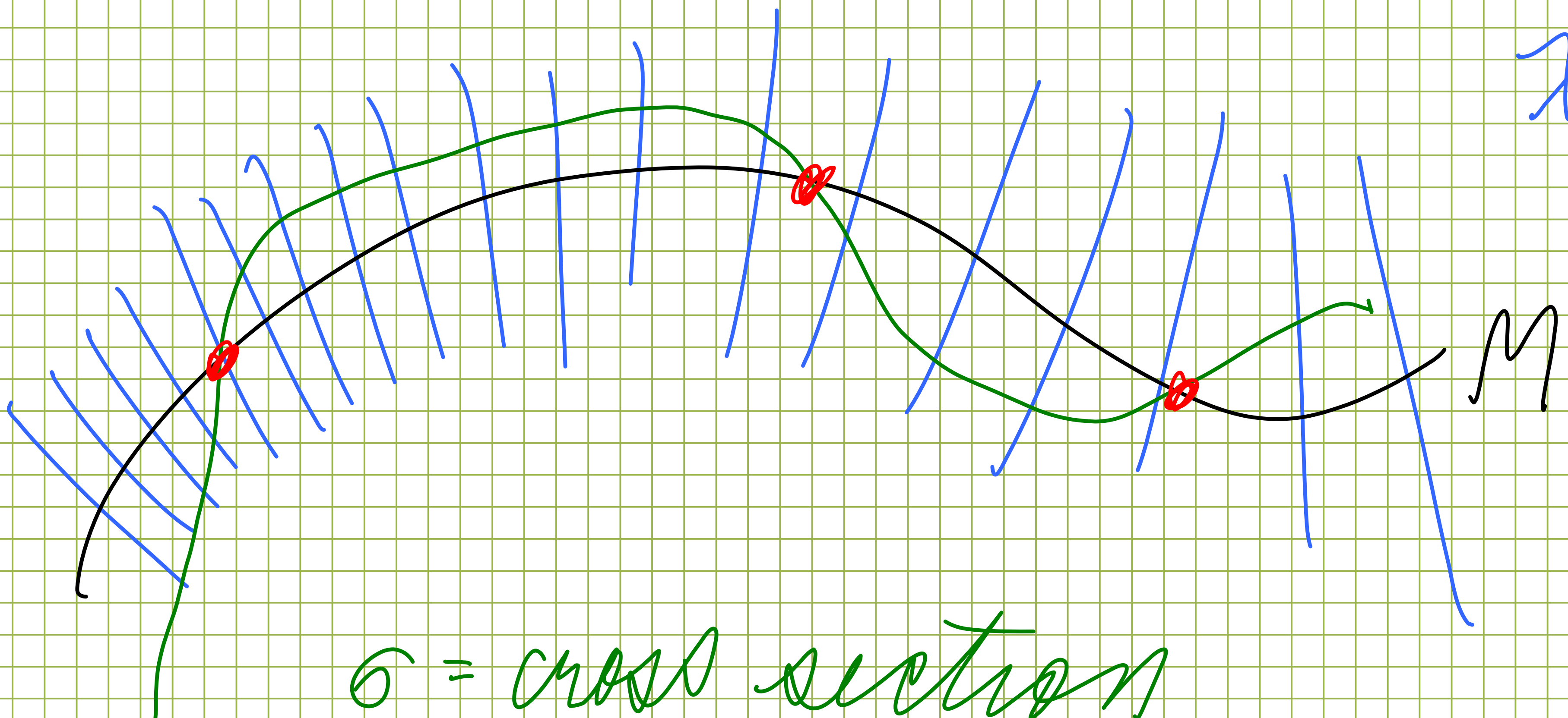
$\lim \Omega^{2i} MU(i+1)$, which defines an elt in $MU^2(M)$.

This is the Conner-Floyd Chern

class $c_1(\lambda)$. The submanifold $N^k \to M$ represents an element in $MU_{2(n-k)}(M)$. There is a Poincaré duality iso $MU^{2k}(M) \cong MU_{2n-2k}(M)$. $[N^k]$ is dual to $c_1(\lambda)^k$.

<u>Remark</u> If M is a complex projective variety and λ is a holomorphic line bundle, then

the set of holomorphic sections of λ is a finite dimensional vector space $V/\mathbb{C}$. The Riemann-Roch thm determines its dimension. There is an open dense subset $W \subset V$ s.t. for $\sigma \in W$ the intersection of $\sigma(M)$ with the 0-section is a submanifold N^1 as above.



σ = cross section

M → total space of λ

Given k sections $\sigma_1, \sigma_2, \cdots \sigma_k$, each leads to a codim 2 subbundle and their intersection is N^k as above.

Example $M = \mathbb{C}P^n = \{[z_0, z_1, \dots, z_n]\}$

ζ = tautological line bundle

$\lambda = \zeta^{\otimes d}$. A homogeneous polynomial P of degree d in the $n+1$ variables $z_0 \dots z_n$ defines a section σ of λ. Its intersection with the 0-section is the variety defined by the equation

$P(z_0, \dots z_n) = 0.$

Suppose we have 2 line bundles λ_1 and λ_2. Each leads to a collection of submfds as above; and we can assume

$$N_1^i \cap N_2^j = \text{submfd of codim } 2(i+j).$$

Its bordism class in $MU_{2(n-i-j)}(M)$ is dual to $c_1(\lambda_1)^i c_1(\lambda_2)^j$

Consider the bundle $\lambda_1 \otimes \lambda_2 = \lambda_3$

$$c_1(\lambda_3) = c_1(\lambda_1 \otimes \lambda_2)$$
$$= \sum_{i,j} a_{ij}\, c_1(\lambda_1)^i\, c_1(\lambda_2)^j \in MU^2(M)$$

where $a_{ij} \in \pi_{2(i+j-1)} MU$.

It is Poincaré dual to

$$\sum_{i,j} a_{ij} \left(N_1^i \cap N_2^j\right) \in MU_{2n-2} M$$

This is the bordism class of N_3^1.

This is the geometric interpretation

of the FGL.

<u>Example</u> $M = \mathbb{C}P^m \times \mathbb{C}P^n$

λ_1 = taut line bundle over $\mathbb{C}P^m$

λ_2 = " $\mathbb{C}P^n$

$N_1^i \cap N_2^j = \mathbb{C}P^{m-i} \times \mathbb{C}P^{n-j} \subset M.$

$\lambda_3 = \lambda_1 \otimes \lambda_2$. A section of it is a

$$P(x_0, \ldots x_m, y_0, \ldots y_n)$$

$$= \sum_{\substack{0 \le s \le m \\ 0 \le t \le n}} c_{s,t}\, x_s\, y_t \qquad c_{s,t} \in \mathbb{C}$$

The resulting submanifold is called the <u>Milnor hypersurface</u>.

$H^{m,n} \subset \mathbb{C}P^m \times \mathbb{C}P^n$

The geometric interpretation of the FGL leads to following formula in $\pi_* MU$ = complex cobordism ring.

Let $P(u) = \sum_{n \geq 0} [\mathbb{C}P^n] u^n$

$$H(u, v) = \sum_{m,n \geq 0} [H^{m,n}] u^m v^n$$

$$F(u,v) = \sum_{m,n\geq 0} a_{m,n} u^m v^n$$

Then (Quillen's formula)

$$H(u,v) = P(u)\,P(v)\,F(u,v) \in \pi_* MU[[u,v]].$$

Milnor showed that the classes $[H^{m,n}]$ generated $\pi_* MU$ as a ring.

Mischenko showed

$$P(u) = 1 / F_2(u,0)$$

where $F_2(u,v) = \dfrac{\partial F(u,v)}{\partial v}$

$$= \frac{\partial}{\partial v} \sum a_{m,n} u^m v^n$$

$$= \sum_{m,n \geq 0} n \, a_{m,n} u^m v^{n-1}$$

$$F_2(u,0) = \sum a_{m,1} u^m$$