Note Title 11/16/2012

Recall we have description of slices for $MU_{\mathbb{R}}$, modulo identifying each slice as a certain wedge of $S^{n\rho} \wedge H\underline{\mathbb{Z}}$. Will deal with this later.

The norm construction

Let $H \subset G$ be a subgroup of index n.

N_H^G : H-spectra $\longrightarrow$ G-spectra

H-spaces $\longrightarrow$ G-spaces.

Example ① $H = e$, X = space $|G| = n$

$\mathrm{Map}(G, X)$ is the space X^n (n-fold Cartesian power of X) with G acting by permuting n-ples.

② H arbitrary, X an H-space

$N_H^G X = \mathrm{Map}_H(G, X)$ (H-equivariant maps $G \to X$) is X^n with G permuting the H-invariant factors.

③ $X = V$ = rep of H

$N_H^G V = \mathrm{ind}_H^G V$ = induced rep of G

$= \bigoplus^n V$

There is an analogous construction on spectra. For an H-spectrum X, $N_H^G X$ is a G-spectrum underlain by $X^{(n)}$ = n-fold smash power of X etc.

How to describe this:

* The category of spectra $\mathcal{S}$ is the category of top. pointed spaces $\mathcal{T}_0$ and formally invert suspension i.e. $(S^1 \wedge -)$

The spectrum $X = \{ X_n : \Sigma X_n \to X_{n+1} \}$ corresponds to $\underrightarrow{\mathrm{holim}}\ S^{-n} \wedge X_n$.

* The category of G-spectra $\mathcal{S}^G$ is obtained from that of pointed G-spaces $\mathcal{T}_0^G$ by formally inverting $S^V \wedge -$ for each findimrep V of G.

Remark: For G finite it suffices to invert $S^{\rho_G} \wedge -$

A G-spectrum X is determined by the data $\left\{ \Sigma^{\rho_G} X_{n\rho_G} \longrightarrow X_{(n+1)\rho_G} \right\}_{n \geq 0}$

The corresponding "space" is

$$\underset{\substack{\longrightarrow \\ n}}{\mathrm{holim}}\ \Sigma^{-n\rho_G} X_{n\rho_G}.$$

Given an H-spectrum

$$X = \underset{\substack{\longrightarrow \\ n}}{\mathrm{holim}}\ S^{-n\rho_H} \wedge X_{n\rho_H}$$

$$N_H^G X = \operatorname*{holim}_{\overrightarrow{n}} S^{-n\rho_G} \wedge N_H^G(X_{n\rho_H})$$

For a rep V of H,

$$N_H^G S^V = S^{\mathrm{Ind}_H^G V}$$ EXERCISE

For a G-spectrum X as above

$$X = \operatorname*{holim}_{\overrightarrow{n}} S^{-n\rho_G} \wedge X_{n\rho_G},$$

$\Phi^G X$ is the spectrum

$$\Phi^G X = \operatorname*{holim}_{\overrightarrow{}} S^{-n} \wedge (X_{n\rho_G})^G$$

For an H-spectrum X we have

$$X = \underrightarrow{\mathrm{holim}}\; S^{-n\rho_H} \wedge X_{n\rho_H}$$

$$N_H^G X = \underrightarrow{\mathrm{holim}}\; S^{-n\rho_G} \wedge N_H^G X_{n\rho_H}$$

$$\underline{\Phi}^G N_H^G X = \underrightarrow{\mathrm{holim}}\; S^{-n} \wedge \left(N_H^G X_{n\rho_H}\right)^G$$

What is that fixed pt space?

For any H-space Y,

$$\left(N_H^G Y\right)^G = \left(\mathrm{Map}_H(G, Y)\right)^G$$

$n = [G:H]$

$$= \left(Y^n\right)^G = Y^H$$

e.g. $\left(N_H^G X_{n\rho_H}\right)^G = X_{n\rho_H}^H$, so

$$\Phi^G N_H^G X = \underrightarrow{\mathrm{holim}}\, S^{-n\rho_H} \wedge X_{n\rho_H}^H$$

$$= \Phi^H X$$

e.g. $H = C_2$, $X = MU_{\mathbb{R}}$, $G = C_{2^{j+1}}$

so $n = 2^j$.

We will write $N_2^{2^{j+1}}$ for N_H^G

$\Phi^G N_2^{2^{j+1}} MU_{\mathbb{R}} \simeq \Phi^H MU_{\mathbb{R}} = MO$.

Recall that for $G = C_2$,

$\pi_\star \Phi^G X = \mathbb{Z}$-graded of $a_\sigma^{-1} \underline{\pi}_\star X(G/G)$

The same argument works for $G = C_{2^{j+1}}$ where σ is the sign rep of G.

This leads to a slice differentials theorem for the slice SS for $N_2^{2^{j+1}} MU_{\mathbb{R}}$.

How to determine the slices for $MU_{\mathbb{R}}$ and its norms.

<u>Def</u> Let X be a G-spectrum and suppose $\pi_k^u X$ is free abelian of rank d. A <u>refinement</u> of it is an equivariant is a G-map

$$W \xrightarrow{f} X$$

where W is a wedge of k-dimensional slice cells such W is underlain by $\bigvee_d S^k$ and the map

$\pi_k^u W \longrightarrow \pi_k^u X$ is an isomorphism.

<u>Example</u> $X = MU_{\mathbb{R}}$, $G = C_2$

$\pi_\star^u MU = \mathbb{Z}[\bar{x}_1, \bar{x}_2, \ldots]$ $\bar{x}_i \in \pi_{2i}$

and each $\bar{x}_i$ corresponds to a G-map $S^{i\rho} \xrightarrow{\bar{x}_i} MU_{\mathbb{R}}$

$S^{i\rho}$ is a $(2i)$ dimensional slice cell. $\pi_2^u MU = \mathbb{Z}\{\bar{x}_1\}$ is refined

$$S^\rho \xrightarrow{\bar{x}_1} MU_{\mathbb{R}}$$

$$\pi_4^u MU = \mathbb{Z}\{\bar{x}_1^2, \bar{x}_2\}$$

and
$$S^{2p} \xrightarrow{\bar{x}_1^2} MU_{\mathbb{R}}$$
$$S^{2p} \xrightarrow{\bar{x}_2} MU_{\mathbb{R}}$$

$$W = S^{2p} \vee S^{2p} \xrightarrow{\bar{x}_1^2 \vee \bar{x}_2} MU_{\mathbb{R}}$$

refines $\pi_4^u MU_{\mathbb{R}} = \pi_4 MU$.