Note Title 11/19/2012

Heading toward a proof of the Slice Theorem, which identifies the slices of $MU_{\mathbb{R}}$ and $N_2^{\Sigma^{n+1}}MU_{\mathbb{R}}$ and, by implication, $K_{\mathbb{R}}$.

Recall the notion of refinement:

X is a G-spectrum

$\underline{\pi}_d^{u} X$ is free abelian

A refinement is a G-map

$$W \xrightarrow{f} X$$

where W is wedge of n-dimensional slice cells and $\pi_d^{u}(f)$ is an isomorphism.

e.g. $G = C_2$, $X = MU_{\mathbb{R}}$. each $\pi^{u}_{2d} MU_{\mathbb{R}} = \pi_{2d} MU$ can be refined by a map from $W = V\ S^{d\rho}$ ρ = regular rep of G.

e.g. $G = C_8$, $X = N_2^8 MU_{\mathbb{R}}$. X is underlain by $MU^{(4)}$.

$$\pi_* MU^{(4)} = \mathbb{Z}[x_{i,j} : i > 0,\ 0 \le j \le 3]$$

$$x_{i,j} \in \pi_{2i}$$

The 4 generators are acted on by $G \ni \gamma$ = generator by

$$x_{i,0} \longmapsto x_{i,1} \longmapsto x_{i,2} \longmapsto x_{i,3} \longmapsto (-1)^i x_{i,0}$$

Let $H = C_2 \subset C_8$

$\pi_0^u X = \mathbb{Z}$ is refined by $S^0 \to X$

$\pi_2^u X = \mathbb{Z}\{x_{1,j}\}$ is refined by

$$\begin{array}{ccc} G_4 \times_H S^{\rho_2} & \longrightarrow & X \\ \wr\big\downarrow & \| & \\ y=0 & S^2 & \end{array}$$

ρ_2 = reg rep of $H = C_2$

Let $x_{1,4} = x_{1,0}$

$\underline{\pi}_4^u X$ is free abelian gp of rank 14 generated by

$$\left\{ \overset{4}{x_{1,j}^2},\ \overset{4}{x_{2,j}},\ \overset{4}{x_{1,j}\, x_{1,j+1}},\ \overset{2}{x_{1,j}\, x_{1,j+2}} \right\}$$

$$\subset \mathbb{Z}[x_{1,j}, x_{2,j}, \dots]$$

This set of gens has 4 orbits (up to sign). The refinement is from

$$W = 3\left(G_+ \wedge_H S^{2\rho_2}\right) \vee \left(G_+ \underset{C_4}{\wedge} S^{\rho_4}\right)$$

ρ_4 = reg rep of C_4

$\pi_8^u X \ni \chi_{1,0}\ \chi_{1,1}\ \chi_{1,2}\ \chi_{1,3}$ which are fixed by G and is refined by a map from S^{ρ_8} where ρ_8 = reg rep of C_8

$\pi_{2d} X$ can be similarly refined.

There is <u>never</u> a summand

of the form $G_{\mathbb{Z}_4} \curvearrowright S^{2d}$

A slicker way to describe this.

For a gp H, let V be a rep of H

so we have S^V. Let

$$S^0[S^V] = S^0 \vee S^V \vee S^{2V} \vee S^{3V} \vee S^{4V} \cdots\cdots$$

$\simeq$ "polynomial algebra on S^V."

Can similarly define

$$S^0[S^{V_1}, S^{V_2}, S^{V_3} \cdots\cdots] = Y$$

Let $x_1 : S^{V_i} \longrightarrow Y$ be the given inclusion $\quad Y = S^0[x_1, x_2, \ldots]$

$$N_H^G Y = S^0[G\cdot x_1, G\cdot x_2, G\cdot x_3, \ldots]$$

where $G\cdot x_i$ is short hand for

$$G_+ \underset{H}{\wedge} S^{V_i}$$

Then $\pi_*^u MU_{\mathbb{R}}$ is refined by a map from $S^0[S^{\rho_2}, S^{2\rho_2}, S^{3\rho_2}, \ldots]$

$$= S^0[\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots] =: W$$

where $\bar{x}_1 : S^{i\rho} \longrightarrow MU_{\mathbb{R}}$ refines

$$S^{2i} \xrightarrow{x_i} MU \qquad \pi_{2i}$$

Similarly for $G = C_{2^{n+1}}$ and $H = C_2$,

$\pi_*^u N_H^G MU_{\mathbb{R}}$ is refined by a map from $A = N_H^G W$.

This A is a G-equivariant ring spectrum over which $N_H^G MU_{\mathbb{R}}$ is a module spectrum.

In the case $G = H = C_2$ we are

saying $MU_{\mathbb{R}}$ is a module over W and the map $\pi_*^u W \longrightarrow \pi_*^u MU_{\mathbb{R}}$ is onto. We can define a spectrum

$$MU_{\mathbb{R}} \underset{W}{\wedge} S^0 \quad \text{"tensor product over } W\text{"}$$

Both $MU_{\mathbb{R}}$ and S^0 are W-modules. The map $W \longrightarrow S^0$ is projection onto the wedge summand in dimension 0.

$W = S^0[\bar{x}_1, \bar{x}_2, \dots]$. The tensor product is defined as follows.

Let $W_n = S^0[\bar{x}_1, \bar{x}_2, \dots \bar{x}_n]$

$MU_{\mathbb{R}}$ and S^0 are W_n-modules

$MU_{\mathbb{R}} \wedge_{W_1} S^0$ is the quotient of $MU_{\mathbb{R}}$ obtained by

$$S^\rho \xrightarrow{\bar{x}_1} MU_{\mathbb{R}}$$

$$\Sigma^\rho MU_{\mathbb{R}} \longrightarrow MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \longrightarrow MU_{\mathbb{R}}$$

The cofiber of this composite is $MU_{\mathbb{R}} \wedge_{W_1} S^0$

$$S^{2\rho} \xrightarrow{\bar{x}_2} MU_{\mathbb{R}} \longrightarrow MU_{\mathbb{R}} \underset{W_1}{\wedge} S^0 = \text{ring}$$

$$\Sigma^{2\rho} MU_{\mathbb{R}} \underset{W_1}{\wedge} S^0 \longrightarrow \left(MU_{\mathbb{R}} \underset{W_1}{\wedge} S^0\right)^{(2)} \longrightarrow MU_{\mathbb{R}} \underset{W_1}{\wedge} S^0$$

The cofiber is $MU_{\mathbb{R}} \underset{W_2}{\wedge} S^0$.

Can define $MU_{\mathbb{R}} \underset{W_n}{\wedge} S^0$

and $MU_{\mathbb{R}} \underset{W}{\wedge} S^0 = R(\infty)$ similarly

$R(\infty)$ is underlain by $H\mathbb{Z}$.