Note Title 11/2/2012

Object: Prove the slice differential theorem for K, which said in certain d_3 is nontrivial.
Will derive it from a more general theorem about the slice SS for $MU_{\mathbb{R}}$, real complex cobordism theory.

$G = C_2$. Will define G-spectrum $MU_{\mathbb{R}}$.

$U(n)$ = gp of $n \times n$ unitary matrices / $\mathbb{C}$
G acts by conjugation with
$U(n)^G = O(n)$ = gp of $n \times n$ orthogonal

matrices $/\mathbb{R}$.

$U(n)$ has a classifying $BU(n)$, the space of complex n-planes in $\mathbb{C}^\infty$

It has a $\mathbb{C}^n$-bundle $\gamma_n^{\mathbb{C}}$ with a universal property.

G acts on everything with fixed pts

$BO(n)$ = space of real n-planes in $\mathbb{R}^\infty$ with an $\mathbb{R}^n$-bundle $\gamma_n^{\mathbb{R}}$ with a universal property.

In both cases there is are corresponding unit disk and sphere bundles, and can collapse the sphere

bundle to a point and get
Thom spaces $MU(n)$ and $MO(n)$
with $MU(n)^G \simeq MO(n)$

We use these Thom spaces to build spectra

MU (complex case ignoring G-action)

$MU_{\mathbb{R}}$ (" with)

MO (real case)

e.g. for a real vector space V of dim n

$MO_V = MO(n)$

for $V = n\rho$ (real rep of G)

$(MU_{\mathbb{R}})_V = MU(n)$ as a G-space.

The spectra MO and MU are very well understood

MO was studied by Thom ~ 1954

$$\pi_*(MO) = \mathbb{Z}/2[y_2, y_4, y_5, y_6, y_8 \cdots]$$

$$= \mathbb{Z}/2[y_i : i \neq 2^n - 1] \quad \dim y_i = i$$

MU was studied by Milnor + Novikov ~ 1960

$$\pi_*(MU) = \mathbb{Z}[x_1, x_2, \cdots] \quad \dim x_i = 2i.$$

Each x_i is represented by a i-dimensional complex mfd.

more precisely by a smooth complex projective algebraic variety defined over $\mathbb{R}$, i.e.

X_i is a subset X_i of some $\mathbb{C}P^m$ $(m \geq i)$ defined by polynomials / $\mathbb{R}$.

X_i has a G-action by conjugation

$X_i^G = Y_i$ is an i-dimensional smooth mfd representing $Y_i \in \Pi_i MO$

and if $i = 2^n - 1$, then Y_i is a boundary. This was known 50 years ago.

All Chern #s of X_{2^n-1} are divisible by 2. This implies all all Stiefel-Whitney #s of Y_{2^n-1} are 0.

Want to study the slice SS for $E = MU_{\mathbb{R}}$.

$$S^{i\rho} \xrightarrow{x_i} E$$

$$x_i \in \underline{\pi}_{i\rho} E(G/G)$$

$$\mathrm{Res}_1^2(x_i) \in \pi_{i\rho} E(G/e) = \pi_{2i}^u E = \pi_{2i} MU$$

= Milnor / Novikov generators

Slice Theorem for $MU_{\mathbb{R}}$ (DEEP) Hu-Kriz

$$P_k^k MU_{\mathbb{R}} = \begin{cases} * & \text{for } k<0 \text{ or } k \text{ odd} \\ \bigvee S^{m\rho} \wedge H\underline{\mathbb{Z}} & \text{for } k = 2m \end{cases}$$

where the # of wedge summands is the # of monomials in the x_i in dim $2m$.

Recall we have elements in $E_2^{*,\bigstar}(MU_{\mathbb{R}})$

$$a = a_\sigma \in \pi_{-\sigma} S^0(G/G) \to \pi_{-\sigma} H\underline{\mathbb{Z}}(G/G) \to \underline{E}_2^{1,1-\sigma}(G/G)$$

$$u_\sigma \in \underline{\pi}_{1-\sigma} H\underline{\mathbb{Z}}(G/e) = \underline{E}_2^{0,1-\sigma}(G/G)$$

$\mathrm{Res}_1^2(a) = 0$

$$u = u_{2\sigma} \in \underline{\pi}_{2-\sigma} H\underline{\mathbb{Z}}(G/G) = \underline{E}_2^{0,2-2\sigma}(G/G)$$

with $\mathrm{Res}_1^2 u_{2\sigma} = u_\sigma^2$

$$\chi_i \in \underline{E}_2^{0,i+i\sigma}(G/G)$$

$E_2^{*,\star}$ is generated as a ring by these elements.

$$E_2^{*,\star}(G/e) = \mathbb{Z}[\mathrm{Res}_1^2(\chi_i), u_\sigma]$$

$$\bar{\chi}_i = u_\sigma^i \chi_i \in \underline{E}_2^{0,2i}(G/e)$$

$\underline{E}_2^{*,*}(G/e) = \mathbb{Z}[\bar{\chi}_i]$ with no differentials

$$\underline{E}_2^{*,\star}(G/G) = \mathbb{Z}[u, a, \chi_i : 2a = 0]$$

Let $N = (n_1, n_2, n_3, \cdots)$ $\quad n_i \geq 0$

$\sum n_i < \infty$

$$x^N = x_1^{n_1} x_2^{n_2} \cdots \cdots \in \pi_{\|N\|\rho}(G/G) = \underline{E}_2^{0,\|N\|\rho}(G/G)$$

where $\|N\| = n_1 + 2n_2 + 3n_3 + \cdots$

$$a^k u^m x^N \in \underline{E}_2^{k,\,(k+2m)(1-\sigma)+\|N\|(1+\sigma)}(G/G)$$

$\underline{E}_2^{*,*}$ is spanned by

$$\{a^k u^m x^N : k+2m = \|N\|\}$$