Note Title 11/28/2012

Reduction Theorem (and Slice Thm) boils down to showing

$$\pi_* \Phi^G R(\infty) \longrightarrow \pi_* \Phi^G H\underline{\mathbb{Z}}$$

is an isomorphism.

Prop 7.5 (i) $\pi_* \Phi^G H\underline{\mathbb{Z}} = \mathbb{Z}/2[b]$ where $b \in \pi_2$

(ii) $\pi_n \Phi^G R(\infty) = \begin{cases} \mathbb{Z}/2 & \text{for } n \geq 0 \text{ even} \\ 0 & \text{otherwise.} \end{cases}$

Proof (i) Let $\sigma$ denote the sign rep of $G$.
We have $a_\sigma \in \pi_{-\sigma} H\underline{\mathbb{Z}}(G/G)$

$$u_{2\sigma} \in \pi_{2-2\sigma} H\underline{\mathbb{Z}}(G/G)$$

$\pi_* \Phi^G H\underline{\mathbb{Z}}$ = the $\mathbb{Z}$-graded part of

$$a_\sigma^{-1} \underline{\pi}_\star H\underline{\mathbb{Z}}(G/G)$$

Let $b = a_\sigma^{-2} u_{2\sigma} \in \underline{\pi}_2 H\underline{\mathbb{Z}}(G/G)$

Previous calculations show

$\pi_* \Phi^G H\underline{\mathbb{Z}} = \mathbb{Z}/2[b]$ as claimed.

(ii)

Recall we have elements

$$\overline{m}_i \in \pi_{i\rho_2} i_2^{*}(MU^{((G))})$$

$$x_j' = N_2^{2^n} \overline{m}_j \in \pi_{jP_G} MU^{((G))}$$

$$S^{jP_G} \xrightarrow{x_j'} MU^{((G))}$$

Apply the functor $\Phi^G$ and get

$$S^j \xrightarrow{\Phi^G x_j'} MO$$

$$\pi_* MO = \mathbb{Z}/2[h_2, h_4, h_5, \ldots] \qquad h_j \in \pi_j$$

$$= \mathbb{Z}/2[h_j : j \neq 2^k - 1, j > 0]$$

The $\overline{m}_j$ can be chosen so that

$$z_j = \Phi^G N_2^{2^n} \overline{m}_j = \begin{cases} 0 & \text{for } j = 2^k - 1 \\ h_j & \text{otherwise} \end{cases}$$

# Notation

$M = MU^{((G))}$ $\qquad C_k = 2^k - 1$

$M_j = MU^{((G))} / (G \circ \bar{r}_j)$

$M_{(k)} = M_{2^k - 1}$

$B = S^0[S^{\rho_2}, S^{2\rho_2}, S^{3\rho_2} \ldots] = C_2$-spectrum

$B \xrightarrow{f} i_2^{2^n} M$ defined by the $\bar{r}_j$

$A = N_2^{2^n} B = S^0[G \circ \bar{r}_1, G \cdot r_2, \ldots]$

$N_2^{2^n} f$ refines $\pi_*^u M$,

$R(\infty) = M \wedge_A S^0$, $\quad B_m = S^0[S^{\rho_2}, \ldots S^{m\rho_2}]$

$A_m = N_2^{2^n} B_m$

$$R(m) = M \wedge_{A_m} S^0$$

$$= MU^{((G))} \Big/ \left(\begin{matrix}\text{first } m \\ \text{sets of generators}\end{matrix}\right)$$

$$\Phi^{G} R(\infty) = \Phi^{G}\left(M \wedge_{A} S^0\right)$$

$$= \Phi^{G} M \wedge_{\Phi^{G} A} \Phi^{G} S^0$$

$$= MO \wedge_{\Phi^{G} A} S^0 = MO \wedge_{\Phi^{C_2} B} S^0$$

and $\Phi^{C_2} B = S^0[S^1, S^2, S^3, \ldots]$

Similarly $\Phi^{G} R(m) = MO \wedge_{\Phi^{C_2} B_m} S^0$

where $\Phi^{C_2} B_m = S^0[S^1, S^2, \dots S^m]$

For each $m$ there is a cofiber ~~sequence~~

$$\Sigma^m \Phi^{G} R(m-1) \xrightarrow{z_m} \Phi^{G} R(m-1) \longrightarrow \Phi^{G} R(m)$$

where $z_m$ is as above.

For $m=1$ we have ($z_1 = 0$)

$$\Sigma MO \xrightarrow{0} MO \longrightarrow \underset{MO \vee \Sigma^2 MO}{\overset{\Phi^{G} R(1)}{\parallel}}$$

For $n=2$ we have

$$\begin{tikzcd} \Sigma^2 \Phi^{G} R(1) \arrow[r, "z_2"] \arrow[d, "\simeq"] & \Phi^{G} R(1) \arrow[r] \arrow[d, equal] & \Phi^{G} R(2) \arrow[d, equal] \\ \Sigma^2 MO \wedge (S^0 \vee S^2) & MO \wedge (S^0 \vee S^2) & MO/(h_2) \wedge (S^0 \vee S^2) \end{tikzcd}$$

Similarly

$$\Phi^{G} R(3) = MO/(h_2) \wedge (S^0 \vee S^2) \wedge (S^0 \vee S^4)$$

$$\Phi^{G} R(4) = MO/(h_2, h_4) \wedge \text{same}$$

$$\Phi^{G} R(5) = MO/(h_2, h_4, h_5) \wedge \text{same}$$

and so on.

This leads

$$\Phi^G R(\infty) = H\underline{\mathbb{Z}}/2 \wedge \bigvee_{i\geq 0} S^{2i}$$

This shows $\pi_* \Phi^G R(\infty)$ is as claimed. QED

<u>Lemma 7.7</u> If $\forall k \geq 0$ the class $b^{2^{k-1}} \in \pi_{2^k} \Phi^G H\underline{\mathbb{Z}}$ is in the image of

$$\pi_* \Phi^G M_{(k)} \longrightarrow \pi_* \Phi^G H\underline{\mathbb{Z}}$$

then our map $\pi_* \Phi R(\infty) \to \pi_* \Phi^G H\underline{\mathbb{Z}}$ is an isomorphism.

Proof $R(\infty) = M_1 \underset{M}{\wedge} M_2 \underset{M}{\wedge} M_3 \underset{M}{\wedge} \cdots$

This is analogous to

$$S = \mathbb{Z}[x_1, x_2, \ldots]$$

$$S_n = S/(x_n) = S\text{-module}$$

$$S_1 \otimes_S S_2 \otimes_S S_3 \otimes_S \cdots = S/(x_1, x_2, \ldots) = \mathbb{Z}$$

This description implies that if the $b^{2^{k-1}}$ are in the image of $M_{(k)}$ then these products are in the image of the product above e.g. if $b$ is in image of $\pi_* \Phi^G M_1$

and $b^2$ $\pi_* \Phi^G M_3$

then $b^3$ " $\pi_* \Phi^G (M_1 \wedge_M M_3)$

Any positive integer is uniquely a sum of distinct powers of 2.

QED