Note Title 11/7/2012

We have determined the slice SS for $MU_{\mathbb{R}}$ and we want it for $K_{\mathbb{R}}$. In the former we have $d_3, d_7, d_{15}$ etc. Will show there is map $MU_{\mathbb{R}} \to K_{\mathbb{R}}$ which forces the $d_3$ in the target SS.

CULTURAL EXCURSION.

Will include a thm of Quillen in 1969. In 1966 Conner + Floyd

wrote a book called "The relation of cobordism to K-theory."
They showed that $K^*X$ is functorially determined by $MU^*X$.

Def A *formal gp law* over a ring $R$ is a power series $F(x,y) \in R[[x,y]]$ with

1) $F(x,0) = F(0,x) = x$ IDENTITY

2) $F(y,x) = F(x,y)$ COMMUTATIVITY

3) $F(F(x,y),z) = F(x,F(y,z)) \in R[[x,y,z]]$ ASSOCIATIVITY.

Examples.

a) $x + y$

b) $x + y + xy = F$ ; $1 + F = (1+x)(1+y)$

c) $\frac{x+y}{1-xy} = (x+y)\sum_{n\geq 0}(xy)^n$

d) $G$ = 1-dimensional abelian Lie gp
choose a co-ordinate patch
centered at $e$. $G \times G \xrightarrow{m} G$
Assume $m$ is analytic, i.e. it
has a convergent power
series expansion $m(x,y) \in \mathbb{R}[[x,y]]$
It is a FGL.

In c) not that

$$\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$$
$$= F(\tan(\alpha), \tan(\beta))$$

Construction of the universal FGL:

$$F(x,y) = \sum_{i,j\geq 0} a_{i,j}\, x^i y^j \qquad a_{i,j} \in R$$

Let $L = \mathbb{Z}[a_{i,j}]\,/\,(\text{relations})$ LAZARD 1955.

Then we have a FGL $F^L$ over $L$

Given any FGL $F$ over any ring $R$
$\exists!\ \theta : L \to R$ with $\theta(F^L) = F$.

$a_{i,j} \longmapsto$ coeff of $x^i y^j$ in $F(x,y)$

$L$ can be graded with

$|a_{i,j}| = i+j-1$ or $2(1-i-j)$

If $\dim x = \dim y = -1$ $2$ then

$\sum a_{i,j} x^i y^j$ is homogeneous

of degree $-1$. $2$

<u>Thm</u> $L \cong \mathbb{Z}[u_1, u_2, u_3, \cdots]$

with $|u_i| = i$ $-2i$

One can define a FGL over $\pi_* MU$ as follows.

A complex line bundle $\lambda$ over

a space $X$ has a first Chern class $c_1(\lambda) \in H^2(X)$. For 2 line bundles $\lambda_1$ and $\lambda_2$,

$$c_1(\lambda_1 \otimes \lambda_2) = c_1(\lambda_1) + c_1(\lambda_2).$$

Conner–Floyd showed that one can do the same in

$$MU^2(X) \ni c_1^{CF}(\lambda)$$

$$c_1^{CF}(\lambda_1 \otimes \lambda_2) = F^{MU}\left(c_1^{CF}(\lambda_1), c_1^{CF}(\lambda_2)\right)$$

$$\in MU^*[[c_1^{CF}(\lambda_1), c_1^{CF}(\lambda_2)]]$$

$MU^* = \pi_* MU$ (negatively graded)

This defines a FGL over $MU^*$.
Hence there is a map $\theta : L \longrightarrow MU^*$
which Quillen shows to be
an isomorphism.

Alternate interpretation.

Let $X = \mathbb{C}P^\infty$ = classifying space
for complex line bundles.

$\mathbb{C}P^n$ = space of $\mathbb{C}$ lines thru $0$
in $\mathbb{C}^{n+1}$

Let $E = \{(x, l) \in \mathbb{C}^{n+1} \times \mathbb{C}P^n : x \in l\}$

$$E \xrightarrow{p} \mathbb{C}P^n \qquad p^{-1}(l) = \text{the set of vectors in } l.$$

$$(x, l) \longmapsto l$$

This is the tautological line bundle over $\mathbb{C}P^n$.

Let $\lambda$ be the taut. line bundle over $\mathbb{C}P^\infty$.

$H^* \mathbb{C}P^\infty = \mathbb{Z}[x] \quad x \in H^2 \quad x = c_1(\lambda)$.

$MU^*(\mathbb{C}P^\infty) = MU^*[[c_1^{CF}(\lambda)]]$

Any line bundle over a paracompact $X$ is induced by a map $X \to \mathbb{C}P^\infty$.

e.g. $X = \mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{p_1} \mathbb{C}P^\infty$

$$\begin{tikzcd} \mathbb{C}P^\infty \times \mathbb{C}P^\infty \arrow[r, "p_1"] \arrow[d, "p_2"] & \mathbb{C}P^\infty \\ \mathbb{C}P^\infty & \end{tikzcd}$$

gives 2 line bundles $\lambda_1$ and $\lambda_2$ over $X$. Their tensor product $\lambda_1 \otimes \lambda_2$ is induced by another map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$.