Note Title 12/11/2012

Summary of Kervaire-Milnor

bP_{4m+2} = gp of exotic $(4m+1)$-spheres that bound framed mfds

It is either 0 or $\mathbb{Z}/2$ depending on the following

Kervaire constructed a framed $(4m+2)$-mfd with boundary N^{4m+2} with

1) $\phi(N) \neq 0$

2) ∂N is a top $(4m+1)$-sphere Σ^{4m+1}

Iff Σ^{4m+1} is exotic then $bP_{4m+2} = \mathbb{Z}/2$ and a smooth closed framed mfld M^{4m+2} has $\phi(M) = 0$.

Iff Σ^{4m+1} is the standard sphere, then $bP_{4m+2} = 0$ and $M^{4m+2} = N^{4m+2} \cup_\partial D^{4m+2}$ has $\phi(M) \neq 0$.

<u>Browder's Thm 1969.</u> $\phi(M^{4m+2}) = 0$ and $bP_{4m+2} = \mathbb{Z}/2$ in all cases except possibly $m = 2^{j-1} - 1$ for $j > 0$. We get $\phi(M^{4m+2}) \neq 0$

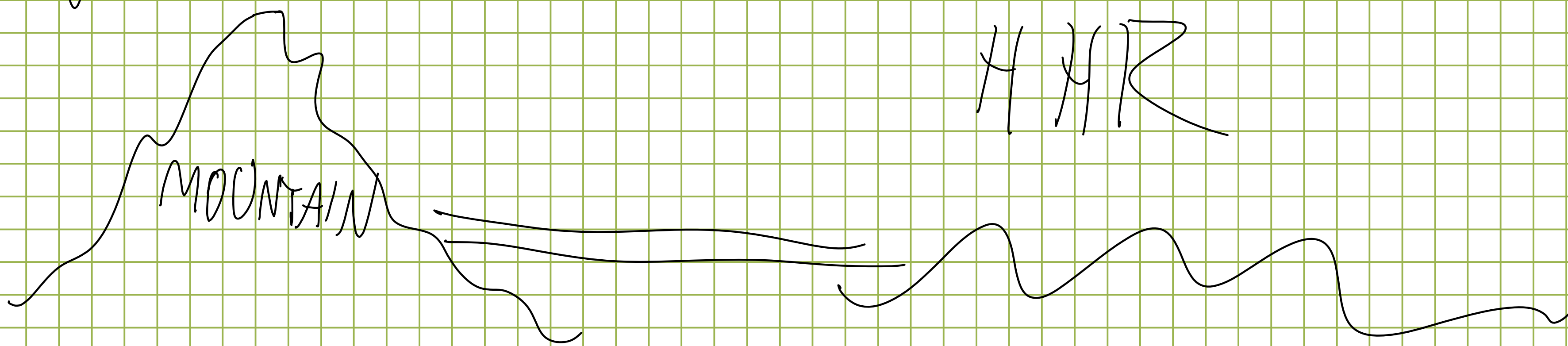
for $m = 2^{j-1}-1$ of h_j^2 in the Adams SS is a permanent cycle.

The corresponding elt in $\pi_{2^{j+1}-2} S^0$ is denoted by θ_j. It is known to exist for $1 \le j \le 5$.

In 1967 Mahowald showed existence of θ_j $\forall j$ has nice consequences.

DOOMSDAY HYPOTHESIS: only finitely many θ_j exist.

Θ_j - problem



Thm HHR 2009 Θ_j does not exist for $j \geq 7$.

Strategy of proof. There is a ring spectrum "magic" = "Omega" with 3 properties

1) Detection Thm. The unit map

$S^0 \to \Omega$ sends θ_j (if it exists) to a nontrivial element in $\pi_* \Omega$

2) <u>Periodicity Thm</u> $\pi_k \Omega$ depends only on k mod 256.

$\Omega^{256} \Omega \simeq \Omega$

3) <u>Gap Thm</u> $\pi_{-2} \Omega = 0$

Now 2) and 3) $\Rightarrow \pi_{254} \Omega = 0$

$254 = \dim \theta_7$ $\therefore$ θ_7 has no nontrivial image in this gp.

How do we construct Ω???

We derived $K_{\mathbb{R}}$ from $MU_{\mathbb{R}}$ by a procedure that involved inventing a class $x \in \pi_\rho MU_{\mathbb{R}}$ and killing some other classes. This required identifying the slices of $MU_{\mathbb{R}}$, which requires the reduction theorem, due in this case to Hu-Kriz 2001.

We find that $\Sigma^8 K_{\mathbb{R}} \simeq K_{\mathbb{R}}$

$K_{\mathbb{R}}$ is a C_2-spectrum

$K_{\mathbb{R}}^{C_2} = K_{\mathbb{R}}^{hC_2} = KO$, the spectrum for real (as opposed to complex) K-theory. If we ignore the C_2-structure, $\Omega^2 K = K$

There is an analogous construction that starts with $N_2^{2^n} MU_{\mathbb{R}}$) which is a C_{2^n}-spectrum.

For each n the slices can be described explicitly. They are wedges of $G_+ \wedge_H S^{mP_H} \wedge H\mathbb{Z}$ for $m \in \mathbb{Z}$ where $G = C_{2^n}$ and H is a nontrivial subgp of G. We can have $m < 0$ after we invert x. We know $\underline{\pi}_{-2}\, G_+ \wedge_H S^{mP_H} \wedge H\mathbb{Z} = 0$. This result will lead us to the gap theorem.

Ω is the C_8-fixed pt set of the telescope derived from

$N_2^8 MV_{\mathbb{R}}$.

$\pi^{-1} N_2^8 MV_{\mathbb{R}} = \Omega$