Summary of Kervaire–Milnor

\[ \mathcal{P}_{4m+2} = \text{gp of exotic } (4m+1) \text{-spheres} \]

that bound framed \(n\)-manifolds

It is either 0 or \(2^1/2\) depending on

the following

Kervaire constructed a framed \((4m+2)\)-

manifold with boundary \(N^{4+m+2}\) with

1) \(\phi(N) \neq 0\)

2) \(\partial N\) is a top \((4m+1)\)-sphere \(\cong 4m+1\)
If $\Sigma^{4m+1}$ is exotic then $b^\mathbb{P}_{4m+2} = \pm 2$

and a smooth closed framed manifold $M^{4m+2}$ has $\phi(M) = 0$.

If $\Sigma^{4m+1}$ is the standard sphere, then $b^\mathbb{P}_{4m+2} = 0$ and $M^{4m+2} = N \cup_2 D^{4m+2}$ has $\phi(M) \neq 0$.

Browder's Thm 1969: $\phi(M^{4m+2}) = 0$

and $b^\mathbb{P}_{4m+2} = \pm 12$ in all cases except possibly $m = 2^{j-1} - 1$

for $j > 0$. We get $\phi(M^{4m+2}) \neq 0$.
for \( m = 2^k - 1 \), \( k \geq 2 \) in the Adams SS is a permanent cycle. The corresponding root in \( \tilde{\mathbb{S}}^{2^k + 1 - 2} \) is denoted by \( \theta_j \). It is known to exist for \( 1 \leq j \leq 5 \).

In 1967 Mahowald showed existence of \( \theta_j \) for \( j \) has nine consequences.

**DOOMSDAY HYPOTHESIS**: only finitely many \( \theta_j \) exist.
Thm: NHR 2009 $Q_j$ does not exist for $j \geq 2$. 

Strategy of Proof: There is a ring spectrum $\Lambda$ with 3 properties: 

1) Detection Thm. The unit map
$g \rightarrow S \rightarrow \Theta$ (if it exists) to a nontrivial element in $T \times S$

2) **Periodicity Thm.** $T \times S$

depends only on $R \bmod 256,$

$2^{256} S \equiv S$

3) **Hopf Thm.** $\Pi_{-2} S \equiv 0$

Now 2) and 3) $\implies \Pi_{254} S \equiv 0$

$254 = \dim \Theta \geq 1$ has no nontrivial image in this $\Theta$!

How do we construct $S \Theta$?
We derived $KR$ from $MVIR$ by a procedure that involved inventing a class $x \in \pi_0 MVIR$ and killing some other classes. This required identifying the slices of $MVIR$, which requires the reduction theorem, due in this case to Hu-Kriz 2001.

We find that $S^k KR \simeq KR$ as $KR$ is a $\mathbb{C}_2$-spectrum.
\[
K_{\mathbb{C}}^+ = K_{\mathbb{R}}^+ = KO \setminus \text{the spectrum for real (as opposed to complex \ K-theory). If we ignore the C_2^* \text{-structure, } \mathbb{Z}^2 \cdot K = K}
\]

There is an analogous construction that starts with \( N_{2^n}^\infty \mathbb{MV}_{\mathbb{R}} \), which is a \( C_{2^{n+1}} \)-spectrum. For each \( n \), the slices can be described explicitly. They are
where \( C_2 = C_2^m \) and it is a nontrivial subgroup of \( C_1 \). We can have \( m < 0 \) after we invert \( x \). We know \( \Pi \rightarrow C_4 \times H \), \( S_{\Pi^m \times H} = 0 \). This result will lead us to the gap theorem.

\( \Omega \) is the \( C_6 \)-fixed but set of the telescope derived from
\[ N_2^8 \text{MUR}^R \]

\[ \chi^{-1} N_2^8 \text{MUR}^R = 52 \]