To identify the classes in $MU^{((G))}$ we need the Reduction Thm ($R(\infty) \xrightarrow{\simeq} H\underline{\mathbb{Z}}$) which has been reduced to the following

Prop 7.14 There is an exact sequence

$$\pi_V^G EP_+ \wedge P_d X \longrightarrow \pi_V^G EP_+ \wedge X \longrightarrow \pi_V^G EP_+ \wedge H\underline{\mathbb{Z}}$$

$$y_R = \mathrm{Im}\, y \longmapsto \neq 0 \,?$$

Suffices to show the above.

Prop 7.15 The image of

underlying

$$
\begin{tikzcd}
 & \pi_V^G P_d X \arrow[dr] & & \\
\pi_V^G EO_+ \wedge P_d X \arrow[ur] \arrow[rr, "\alpha"] & & \pi_V^G X \arrow[r, "i_0^*"] & \pi_d^u X
\end{tikzcd}
$$

$\pi_V^G X \ni y_R$

$\pi_d^u X = \mathbb{Z}[G]$-module

The image of $i_0^*\alpha$ is contained in that of $1-\gamma$ for $\gamma \in G$ a generator.

(to be proved)

We know that $i_0^* y_R$ is <u>not</u> in that image be because $\gamma(i_0^* y_R) = -i_0^* y_R$.

To prove 7.15 we use

<u>Prop 7.16</u> Let Y be a G-spectrum with $Y \geq 0$ (i.e. slice (-1)-connected)

Then the image of

$$\pi_\star^G EP_+ \wedge Y \longrightarrow \pi_\star^G Y$$

is contained in the image of Tr_H^G.

proved last time

Cor 7.17 + 7.18 For $Y = \Sigma^{-V} P_d X$ we get

$$\begin{tikzcd}
\pi_V^G EP_+ \wedge P_d X \arrow[r] & \pi_V^G P_d X \arrow[r] & \pi_V^G X \\
 & \pi_V^H P_d X \arrow[u, "\mathrm{Tr}_H^G"] \arrow[r] & \pi_V^H X \arrow[u, "\mathrm{Tr}_H^G"']
\end{tikzcd}$$

The image of the top composite is contained in that of Tr_H^G

Lemma 7.19 For a rep V with $d = |V|$ and a G-spectrum X (arbitrary V and X), let $\varepsilon := \deg : S^V \xrightarrow{\gamma} S^V$

$= \det(\gamma) = \pm 1$

Then

$$\pi_V^H X \xrightarrow{Tr_H^G} \pi_V^G X \xrightarrow{i_0^*} \pi_d^n X$$

has image contained in that of $1 + \varepsilon\gamma$.

For $V = \mathbb{C}_\mathbb{R}\rho_G$, $\varepsilon = -1$.

Proof: Consider the diagram

$$\begin{tikzcd}
\pi_V^H X \arrow[r, "\cong"] & \pi_V^G\, C_{2+}\wedge X \arrow[r] \arrow[d, "i_0^*"] & \pi_V^G X \arrow[d, "i_0^*"] \\
\pi_d^H X \oplus \pi_d^H X = & \pi_d^H (C_{2+}\wedge X) \arrow[r, "\text{fold}"] & \pi_d^H X
\end{tikzcd}$$

$$C_{2+} = S^0 \vee S^0, \text{ so } \quad C_{2+}\wedge X = X \vee X$$

$$\begin{tikzcd}
X \vee X \arrow[d, "1 \vee \gamma"] \\
X
\end{tikzcd}$$

Left vertical map has a γ-invariant image, so each element has the form $(a, \varepsilon\gamma a)$

$$(a, b) \longmapsto a + b$$

$$\begin{tikzcd}
(a, b) \arrow[d, "\gamma"] \\
(\varepsilon\gamma b, \varepsilon\gamma a)
\end{tikzcd}$$

$$(a, \varepsilon\gamma a) \longmapsto (1+\varepsilon\gamma)a$$

QED

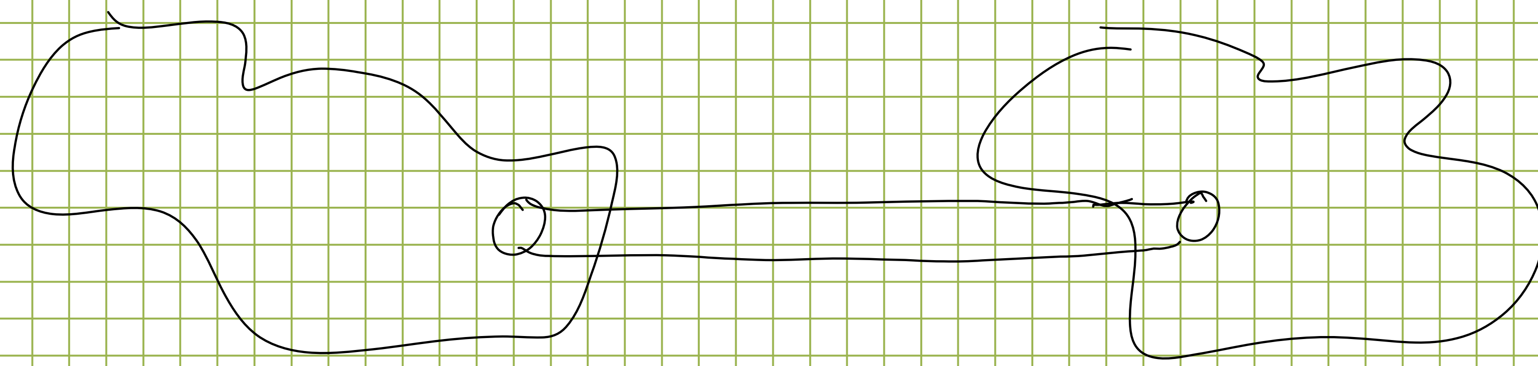
This implies 7, 15 and hence the Reduction Theorem.
This is the technical core of HHR.

What is the Kervaire invariant problem.

In 1956 Milnor showed there are smooth 7-manifold that are homeomorphic but not diffeomorphic to the standard S^7.

In 1960 Kervaire constructed a topological 10-manifold with no differentiable structure.

In 1964 they classified diff'ble structures on S^n for $n \geq 5$ subject to 2 caveats. Let Θ_n be the gp of such structures. (under connected sum + orientation reversal)



M_1 M_2

1) Θ_n is described in terms of

$$\pi_n S^0 = \lim_{k \to \infty} \pi_{n+k} S^k$$

2) For $n \equiv 1 \mod 4$ there is an ambiguous factor of 2 in Θ_n. We now know this

factor in each case except $n=125$.

Outline of Kervaire-Milnor.

Let Σ^n denote a smooth mfd homeomorphic to S^n.

Σ^n can be embedded in some $\mathbb{R}^{n+k}$ with trivial normal bundle.

Let V be a closed tubular nbd of $\Sigma^n \subset \mathbb{R}^{n+k}$ and choose a homeo $V \cong \Sigma^n \times D^k$. This is called a <u>framing</u>. (We can replace Σ^n

by any M^n with the same property.)
Use this to construct a map
$S^{n+k} \longrightarrow S^k$, the
Pontryagin - Thom construction.

$$\begin{tikzcd}
M \arrow[r] & V \arrow[r] \arrow[d] \arrow[rrr, "f"] & \mathbb{R}^{n+k} \arrow[r] & S^{n+k} \arrow[r, "\hat{f}"] & D^k/\partial D^k = S^k \\
& M \times D^k \arrow[r] & D^k \arrow[rr] & & D^k/\partial D^k = S^k \\
& \partial V \arrow[r] & \partial D^k \arrow[rr] & & pt = x_0
\end{tikzcd}$$

f can be extended to S^{n+k} by
sending everything outside V to x_0