MATH 550

Note Title 8/31/2012

Definition Let G be a topological gp and X a top. space. A G-action on X is a continuous map $G \times X \xrightarrow{f} X$ such that

$$\begin{tikzcd}
(g_1, g_2, x) \arrow[r, mapsto] \arrow[d, mapsto] & (g_1 g_2, x) \arrow[d, mapsto] \\
g_1, g_2(x) \arrow[r, mapsto] & (g_1 g_2)(x) = g_1(g_2(x))
\end{tikzcd}
\qquad
\begin{tikzcd}
G \times G \times X \arrow[r, "m \times G"] \arrow[d, "G \times f"'] & G \times X \arrow[d, "f"] \\
G \times X \arrow[r, "f"] & X
\end{tikzcd}$$

commutes.

Let $f(g, x) =: g(x)$ $\quad g \in G$, $x \in X$

e.g. for the identity elt $e \in G$, $e(x) = x$.

Examples ① $G = \mathbb{Z}$ = integers

$X = \mathbb{R}$ = real line

$f(n, x) = n + x$. Action by translation

② $G = S_n$ = symmetric gp on n letters

$X = Y^n$ = n-fold Cartesian product

$X^G = \Delta(Y)$
$= \{(y, y, \cdots y) \in Y^n\}$

G acts on X by permuting co-ordinates.

③ $G = S_n$, X = disjoint union of n copies of Y

G permutes these n copies.

Definition: Given a G-space X and a subgp $H \subset G$, the fixed point set of H,

$$X^H = \{ x \in X : h(x) = x \text{ for all } h \in H \}$$

(It could be empty for $H \neq \{e\}$.) For $H \subset K \subset G$ then we have $X^H \longleftarrow X^K$ a restriction map.

Definition Given $x \in X$, its isotropy gp $G_x = \{ g \in G : g(x) = x \}$

Definition Two points $x, x' \in X$ are in the **same orbit** if $\exists g \in G$ with $g(x) = x'$.

This is an equivalence relation, so we get equivalences classes called orbits. The set of orbits inherits a topology from X. We get an orbit space X/G.

Examples In (1) above, $X/G = S^1$

In (3) above $X/G = Y$

In ② above $X/G = SP^n(Y)$

= n-fold symmetric product of Y.

= space of unordered n-Tuples

For $Y = S^2$, then $SP^n(S^2) \simeq \mathbb{C}P^n$.

EXERCISE.

<u>Example 4</u> : G = finite gp

X = finite set (discrete topology)

If G acts on X, we say X is a <u>finite G-set</u>.

Such things can be classified
The orbit of $x \in X$ has the form
G/G_x = left (right?) cosets of G_x

<u>Example</u> In (2) with Y finite
$Y = \{a, b, c, d\}$ $n = 3$
$X = Y^3$, 64 elements.

$x = (a, a, b)$
orbit $Gx = \{(a,a,b), (a,b,a), (b,a,a)\}$
In general each orbit in a finite

G-set has a conjugacy class of subgps associated with it

<u>Thm</u> The set of of isomorphism classes of finite G-sets is the free abelian monoid on the set of conjugacy classes of subgps H of G.

In the above example

there are 4 orbits of the form G/G

" 12 " G/S_2

0 G/C_3

4

G/e