**Definition** Let $G$ be a group and $X$ a topological space. A $G$-action on $X$ is a map $\theta : G \times X \to X$ such that:

- $\theta : (g_1, g_2, x) \mapsto (g_1 g_2, x)$
- $\theta : (g_1 x, g_2) \mapsto g_1 (g_2 x)$

Let $f : (g, x) \mapsto g(x)$ for the identity elt $e \in G$.

E.g. for the identity elt $e \in G$:

$e(x) = x$. 

Examples

1. $G = \mathbb{Z}$, integers, $X = \mathbb{R} = \text{real line}$
   
   $f(n,x) = n + x$. Action by translation

2. $G = S_n = \text{symmetric group on } n \text{ letters}$
   $X = Y^n = n\text{-fold Cartesian product}$
   $G$ acts on $X$ by permuting coordinates.

3. $G = S_n$, $X = \text{disjoint union of } n \text{ copies of } Y$
   $G$ permutes these $n$ copies.
Definition: Given a $G$-space $X$ and a subset $H \subseteq G$, the fixed point set of $H$ is

\[ X^H = \{ x \in X : h(x) = x \text{ for all } h \in H \} \]

(It could be empty for $H \neq \{0\}$.) For $H \subset K \subset G$, then we have

\[ X^H \subseteq X^K \quad \text{a restriction map.} \]

Definition: Given $x \in X$, its isotropy group $G_x$ is

\[ G_x = \{ g \in G : g(x) = x \} \]
Definition: Two points \( x, x' \in X \) are in the same orbit if \( \exists g \in G \) with \( g(x) = x' \).

This is an equivalence relation, so we get equivalence classes called orbits. The set of orbits inherits a topology from \( X \). We get an orbit space \( X/G \).

Examples: In (i) above, \( X/G = \mathbb{S}^1 \).
In (ii) above, \( X/G = Y \).
In (2) above \( X/G = \mathbb{P}^n(Y) \)

- \( n \)-fold symmetric product of \( Y \).

- Space of unordered \( n \)-tuples.

For \( Y = S^2 \), then \( \mathbb{P}^n(S^2) \cong \mathbb{C}P^n \).

**Exercise.**

**Example 4:** \( G = \text{finite gp} \)

\( X = \text{finite set (discrete topology)} \)

If \( G \) acts on \( X \), we say \( X \) is a

**finite \( G \)-set.**
Each thing can be classified
The orbit of $x \in \mathbb{F}$ has the form
$G / G_{x} = \text{left (right?) orbit of } G_{x}$

**Example** In (2) with $Y$ finite
$Y = \{ a, b, c, d \}$
$n = 3$

$x = Y^{3} \setminus 64 \text{ elements,}$

$x = (a, a, b)$

orbit $G \cdot x = \{ (a, a, b), (a, b, a), (b, a, a) \}$

In general each orbit in a finite
A set has a conjugacy class of subgroups associated with it.

Then the set of isomorphism classes of finite $G$-sets is the free abelian monoid on the set of conjugacy classes of subgroups $H$ of $G$.

In the above example, there are 4 orbits of the form $G/H$:

- $H/\{1\}$
- $H/\{2\}$
- $H/\{0\}$
- $G/\{S_2\}$
- $G/\{C_3\}$