

Recall the Mackey functor (finite G -set \rightarrow abelian gp)
 $\underline{\Pi}_n(X)$ for a G -space X

Given a finite G -set S we define

$$\underline{\Pi}_n X(S) = [S_+ \times S^n, X]_G^*$$

Formal properties

① Additive on disjoint unions

$$\underline{\Pi}_n(X)(S_1 \sqcup S_2) = \underline{\Pi}_n X(S_1) \oplus \underline{\Pi}_n X(S_2)$$

② Contravariant functor

a map $S_1 \rightarrow S_2$ induces

$$\underline{\pi}_n(X)(S_1) \longleftarrow \underline{\pi}_n(X)(S_2)$$

Let $S = G/H$ for a subgroup H ,
(defined up to conjugacy)

$$\underline{\pi}_n X(G/H) = [G/H, S^n, X]^G_*$$

H acts trivially on the source, so
any \setminus equiv map must land in X^H

$$\text{Hence } \underline{\pi}_n X(G/H) = \underline{\pi}_n(X^H)$$

For $H \subset K \subset G$, the map $G/H \rightarrow G/K$

$$\text{induces } \underline{\pi}_n(X^K) \xrightarrow{\text{Res}_H^K} \underline{\pi}_n(X^H)$$

the restriction map.

Given a G -CW structure on X , we
 get a G -cellular chain complex $C(X)$ of
 $\mathbb{Z}[G]$ -modules. We have a functor
 $\mathbb{Z}[G]$ -modules \rightarrow Mackey functors
 $C(X) \rightarrow \underline{C}(X) = \text{chain}$
 $\text{cx of Mackey functors}$
 $H_* (\underline{C}(X)) = \underline{H}_*(X)$
 is a graded Mackey
 functor.

$$\underline{\pi}_n X(G/H) = \pi_n (X^H)$$

$$\underline{H_n} X (G/N) \neq H_n (X^H)$$

What is the meaning of $\underline{H}_* X$???

COMPUTATION PRECEDES THEORY

Example Let V be a Euclidean vector space on which G acts orthogonally.

This is an orthogonal real rep of G .

1) $G = C_2$, $V = \mathbb{R}^n$ G acts via -1

2) $G = C_4$, $V = \mathbb{R}^4$ G acts by permuting an orthonormal basis.

Let $S(V) =$ unit sphere in V $S(V) \cong S^{|V|-1}$

Δ is G -invariant. Let $|V| = \dim V$.
 Let $S^V =$ one point compactification
 of V . $S^V = S^{|V|}$

Example 1) $S(V)$ has a G -CW structure
 with $C_i = \mathbb{Z}[G]$. To get a CW-structure
 on $X = S^V$: $X^0 = 2$ points, 0 and ∞
 on north + south poles.
 an i -cell in $S(V)$ leads to an $(i+1)$ -cell

in $X = S^V$.

$$C_i(X) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } i=0 \\ \mathbb{Z}[G] & \text{for } 0 < i \leq n. \end{cases}$$

The reduced cellular chain cx is

$$\overline{C}_i(X) = \begin{cases} \mathbb{Z} & \text{for } i=0 \\ \mathbb{Z}[G] & \text{for } 0 < i \leq n. \end{cases}$$

What is the boundary operator?

It is a $\mathbb{Z}[G]$ map. Since $X = S^n$

we know $\overline{H}_i(X) = H_i S^n$

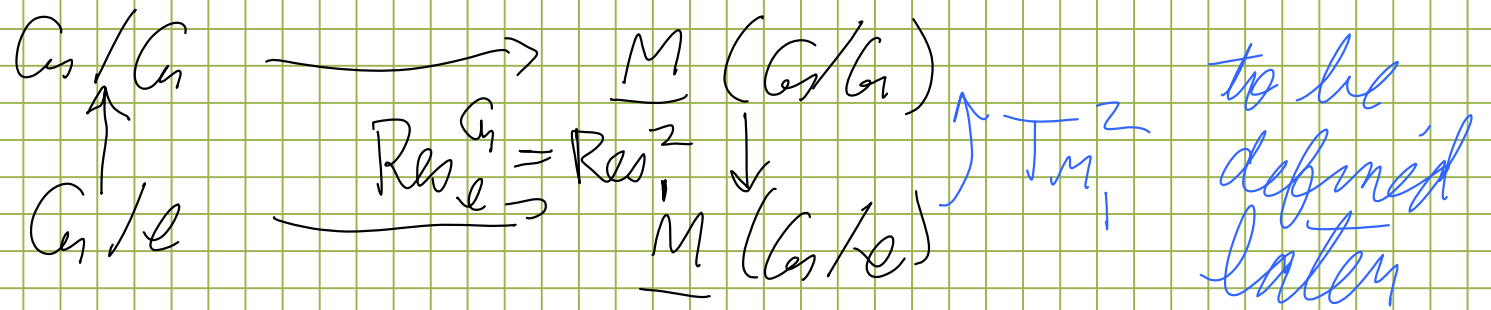
$n=2,3$

$$\begin{array}{ccccccc} \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} \\ \downarrow \scriptstyle 1 & \nearrow \scriptstyle 2 & \downarrow \scriptstyle 1+x & \nearrow \scriptstyle \nabla & \downarrow \scriptstyle 1+x & \nearrow \scriptstyle \nabla & \downarrow \scriptstyle 1+x \\ \mathbb{Z} & \xleftarrow{\nabla} & \mathbb{Z}G & \xleftarrow{1-x} & \mathbb{Z}G & \xleftarrow{1+x} & \mathbb{Z}G \\ 0 & & 1 & & 2 & & 3 \end{array}$$

$x = \text{generator of } G$
 $G = C_2$
 $\mathbb{Z}G = \mathbb{Z}[x]/(x^2-1)$

What is a C_2 Mackey functor?

Smith C_2 -sets \rightsquigarrow abelian gps



Let M be a $\mathbb{Z}G$ -module

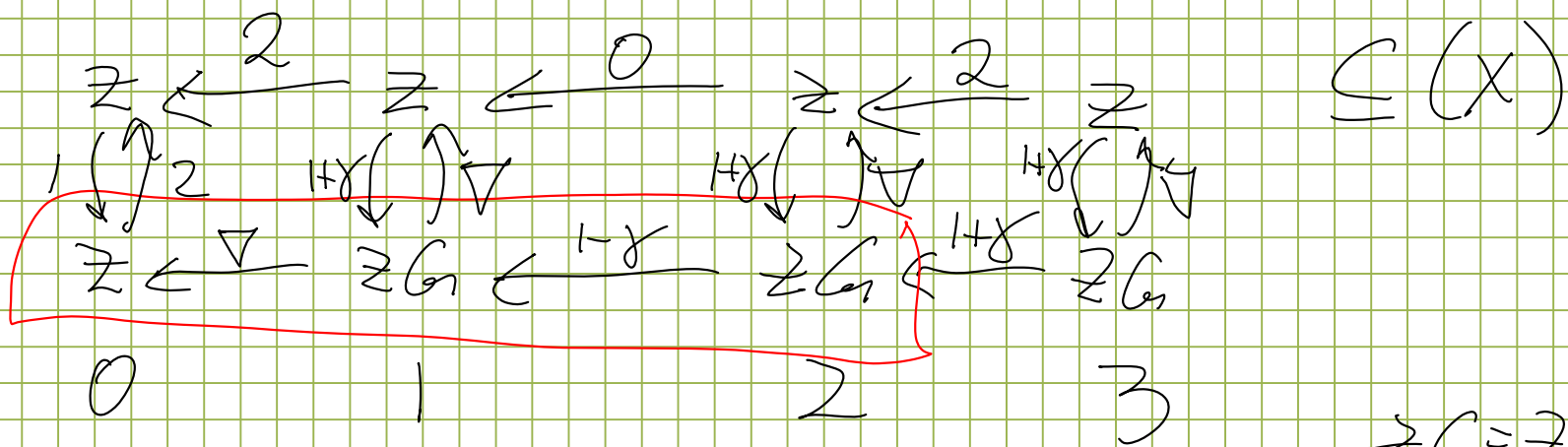
$$\tau_{M,1}^2(m) = (1+\gamma)m$$

$$\underline{M}(G/H) = M^H$$

$$\text{SA } \underline{M}(G/e) = M$$

$$\underline{M}(G/G) = M^G$$

$$\tau_{M,1}^2 m = (1+\gamma)m$$



Homology for $n=2$

$$\mathbb{Z}G = \mathbb{Z}[\gamma] / (\gamma^2 - 1)$$

$$\mathbb{Z} = \mathbb{Z}G / (1 - \gamma)$$

$$\begin{array}{c}
 \mathbb{Z}/2 \\
 \downarrow \uparrow \\
 0
 \end{array}$$

$$\begin{array}{c}
 0 \\
 \downarrow \uparrow \\
 0
 \end{array}$$

$$\begin{array}{c}
 \mathbb{Z} \\
 \downarrow \uparrow 2 \\
 \mathbb{Z}
 \end{array}$$

What happens for $n=3$?