Note Title 9/10/2012

Recall the Mackey functor (finite G-set $\rightsquigarrow$ abelian gp)

$\underline{\pi}_n(X)$ for a G-space X

Given a finite G-set S we define

$$\underline{\pi}_n X(S) = [S_+ \wedge S^n, X]^G_*$$

Formal properties

① Additive on disjoint unions

$$\underline{\pi}_n(X)(S_1 \amalg S_2) = \underline{\pi}_n X(S_1) \oplus \underline{\pi}_n X(S_2)$$

② Contravariant functor

a map $S_1 \longrightarrow S_2$ induces

$$\pi_n(X)(S_1) \longleftarrow \pi_n(X)(S_2)$$

Let $S = G/H$ for a subgp H.
(defined up to conjugacy)

$$\pi_n X(G/H) = [G/H_+ \wedge S^n, X]^G_*$$

H acts trivially on the source, so any equiv map must land in X^H.

Hence $\pi_n X(G/H) = \pi_n(X^H)$

For $H \subset K \subset G$, the map $G/H \to G/K$ induces $\pi_n(X^K) \xrightarrow{\mathrm{Res}^K_H} \pi_n(X^H)$ the restriction map.

Given a G-CW structure on X, we get a G-cellular chain complex $C(X)$ of $\mathbb{Z}[G]$-modules. We have a functor

$$\mathbb{Z}[G]\text{-modules} \longrightarrow \text{Mackey functors}$$

$$C(X) \longrightarrow \underline{C}(X) = \text{chain cx of Mackey functors}$$

$$H_*(\underline{C}(X)) = \underline{H}_*(X)$$

is a graded Mackey functor.

$$\underline{\pi}_n X(G/H) = \pi_n(X^H)$$

$\underline{H}_n X (G/H) \neq H_n (X^H)$

What is the meaning of $\underline{H}_* X$???

COMPUTATION PRECEDES THEORY

<u>Example</u> Let V be a Euclidean vector space on which G acts orthogonally. This is an orthogonal real rep of G.

1) $G = C_2$, $V = \mathbb{R}^n$ G acts via -1

2) $G = C_4$, $V = \mathbb{R}^4$ G acts by permuting an ortho-normal basis.

Let $S(V)$ = unit sphere in V $S(V) \cong S^{|V|-1}$

It is G-invariant. Let $|V| = \dim V$.
Let S^V = one point compactification of V. $S^V \simeq S^{|V|}$

Example 1) $S(V)$ has a G-CW structure with $C_i = \mathbb{Z}[G]$. To get a CW-structure on $X = S^V$: X^0 = 2 points, 0 and ∞ or north + south poles.

an i-cell in $S(V)$ leads to an $(i+1)$-cell in $X = S^V$.

$$C_i(X) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } i = 0 \\ \mathbb{Z}[G] & \text{for } 0 < i \leq n. \end{cases}$$

The reduced cellular chain cx is

$$\overline{C}_i(X) = \begin{cases} \mathbb{Z} & \text{for } i=0 \\ \mathbb{Z}[G] & \text{for } 0<i\le n. \end{cases}$$

What is the boundary operator?
It is a $\mathbb{Z}[G]$-map. Since $X \simeq S^n$
we know $\overline{H}_i(X) = H_i S^n$

γ = generator of G

$n = 2, 3$

$$\begin{tikzcd}
\mathbb{Z} \arrow[d, bend right, "1"'] & \mathbb{Z} \arrow[l, "2"'] \arrow[d, bend right, "1+\gamma"'] & \mathbb{Z} \arrow[l, "0"'] \arrow[d, bend right, "1+\gamma"'] & \mathbb{Z} \arrow[l, "2"'] \arrow[d, bend right, "1+\gamma"'] \\
\mathbb{Z} \arrow[u, bend right, "2"'] & \mathbb{Z}G \arrow[l, "\nabla"'] \arrow[u, bend right, "\nabla"'] & \mathbb{Z}G \arrow[l, "1-\gamma"'] \arrow[u, bend right, "\nabla"'] & \mathbb{Z}G \arrow[l, "1+\gamma"'] \arrow[u, bend right, "\nabla"']
\end{tikzcd}$$

0 $\qquad$ 1 $\qquad$ 2 $\qquad$ 3

$G = C_2$

$\mathbb{Z}G = \mathbb{Z}[\gamma]/(\gamma^2-1)$

What is a C_2 Mackey functor?
finite C_2-sets $\rightsquigarrow$ abelian gps

$$\begin{tikzcd}
G/G \arrow[r] & \underline{M}(G/G) \arrow[d, "\mathrm{Res}^G_e = \mathrm{Res}^2_1"] \\
G/e \arrow[u] \arrow[r] & \underline{M}(G/e) \arrow[u, bend right, "\mathrm{Tr}^2_1"']
\end{tikzcd}$$

to be defined later

Let M be a $\mathbb{Z}G$-module

$\mathrm{Tr}^2_1(m) = (1+\gamma)m$

$\underline{M}(G/H) = M^H$

so $\underline{M}(G/e) = M$ $\qquad$ $\underline{M}(G/G) = M^G$

$\mathrm{Tr}^2_1 m = (1+\gamma)m$

$$\begin{tikzcd}
\mathbb{Z} \arrow[d, shift right, "1"'] & \mathbb{Z} \arrow[l, "2"'] \arrow[d, shift right, "1+\gamma"'] & \mathbb{Z} \arrow[l, "0"'] \arrow[d, shift right, "1+\gamma"'] & \mathbb{Z} \arrow[l, "2"'] \arrow[d, shift right, "1+\gamma"'] \\
\mathbb{Z} \arrow[u, shift right, "2"'] & \mathbb{Z}G \arrow[l, "\nabla"'] \arrow[u, shift right, "\nabla"'] & \mathbb{Z}G \arrow[l, "1-\gamma"'] \arrow[u, shift right, "\nabla"'] & \mathbb{Z}G \arrow[l, "1+\gamma"'] \arrow[u, shift right, "\nabla"']
\end{tikzcd}$$

$$0 \qquad 1 \qquad 2 \qquad 3$$

$$\subseteq (X)$$

$$\mathbb{Z}G = \mathbb{Z}[\gamma]/(\gamma^2 - 1)$$

$$\mathbb{Z} = \mathbb{Z}G/(1-\gamma)$$

Homology for $n=2$

$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[d, shift right] & 0 \arrow[d, shift right] & \mathbb{Z} \arrow[d, shift right, "1"'] \\
0 \arrow[u, shift right] & 0 \arrow[u, shift right] & \mathbb{Z} \arrow[u, shift right, "2"']
\end{tikzcd}$$

What happens for $n = 3$?