Recall Mackey functor chain cx for $X = S^{n\sigma}$ $\qquad G = C_2$

$$\underline{C}(X)$$

$n=2$ or 3

$$\begin{tikzcd}
\mathbb{Z} \arrow[d, shift right, "1"'] & \mathbb{Z} \arrow[l, "2"'] \arrow[d, shift right, "1+\gamma"'] & \mathbb{Z} \arrow[l, "0"'] \arrow[d, shift right, "1+\gamma"'] & \mathbb{Z} \arrow[l, "2"'] \arrow[d, shift right, "1+\gamma"'] \\
\mathbb{Z} \arrow[u, shift right, "2"'] & \mathbb{Z}G \arrow[l, "\nabla"] \arrow[u, shift right, "\nabla"'] & \mathbb{Z}G \arrow[l, "1-\gamma"] \arrow[u, shift right, "\nabla"'] & \mathbb{Z}G \arrow[l, "1+\gamma"] \arrow[u, shift right, "\nabla"']
\end{tikzcd}$$

$0 \qquad 1 \qquad 2 \qquad 3$

σ = sign rep

$\gamma \mapsto [-1]$

$$\gamma(1-\gamma) = \gamma - \gamma^2$$
$$= \gamma - 1$$
$$= -(1-\gamma)$$

Homology for $n=3$

$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[d, shift right] & 0 \arrow[d, shift right] & \mathbb{Z}/2 \arrow[d, shift right] & 0 \arrow[d, shift right] \\
0 \arrow[u, shift right] & 0 \arrow[u, shift right] & 0 \arrow[u, shift right] & \mathbb{Z}_- \arrow[u, shift right]
\end{tikzcd}$$

$\mathbb{Z}_- = \mathbb{Z}G/(1+\gamma)$

Def. A rep V of G is <u>oriented</u> if the matrix associated to each elt in G has det 1.

Example $n\sigma$ is oriented iff n is even

$$\det(-I) = (-1)^n$$

In a Mackey functor $\underline{M}$ for an abelian gp G, $\underline{M}(G/H)$ has an action of the gp G/H.

Remark Recall $\underline{\pi}_i X(G/H) = \pi_i(X^H)$

but in our examples $(n=2,3)$

$$\underline{H}_* X(G/G) \neq H_*(X^G)$$

because $X^G = S^0$ but

$$\underline{H}_\star X(G/G) \neq H_\star S^0$$

What is the meaning of this group???

<u>Another example</u> $G = C_4$, $X = S^{\rho_4}$

ρ_4 = regular rep of C_4
= permutation action on $\mathbb{R}^4$

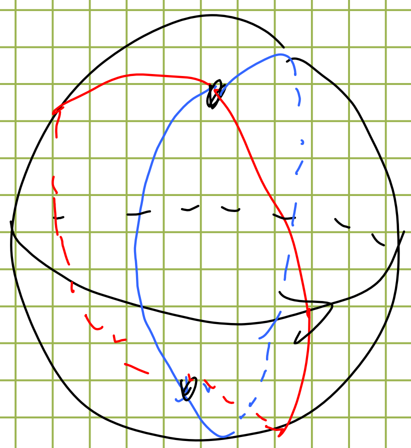
This rep has a trivial 1-dimensional subspace generated by $(1,1,1,1)$

$\overline{\rho_4}$ = reduced regular rep
= subspace $\{(w,x,y,z) : w+x+y+z=0\}$

$S(\overline{\rho_4}) \simeq S^2$. Let γ be a generator of C_4

$$\gamma \longmapsto \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

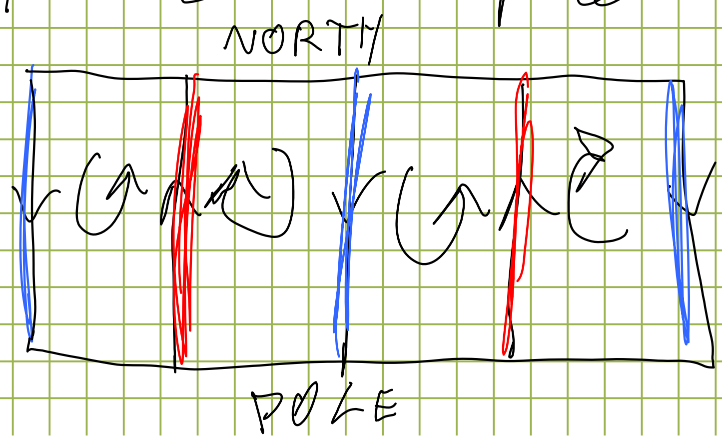
The eigenvector for -1 in $\mathbb{R}^4$ is $(1, -1, 1, -1)$

$Y =$ 

γ reflects thru equator (swapping the poles and rotates around vertical axis by $\pi/2$.

Cellular structure on $Y = S(\bar{P}_4)$

$Y^0 = S^0 = 2$ poles. We have 4 1-cells as shown



and 4 2-cells

G permutes the cells in each dim.

$S^{\overline{\rho}_4}$ will have cells in dims 1, 2, 3 that are double cones on the cells of Y, along with 2 0-cells that are fixed by G.

S^{ρ_4} has a single 0-cell } fixed
1-cell } by G

cells in dims 2, 3, 4 obtained from the cells of Y by joining with S^1.

The resulting reduced cellular chain complex is $\qquad H = C_2 \subset C_4$

$$\mathbb{Z} \xleftarrow{\nabla} \mathbb{Z}G/H \xleftarrow{1-\gamma} \mathbb{Z}G \xleftarrow{1+\gamma} \mathbb{Z}G$$

$$1 \qquad\qquad 2 \qquad\qquad 3 \qquad\qquad 4$$

The homology of this complex is $\overline{H}_* S^4$

H_4 is generated by $\theta = (1-\gamma)(1+\gamma^2) = 1-\gamma+\gamma^2-\gamma^3$

$(1+\gamma)\theta = (1+\gamma)(1-\gamma)(1+\gamma^2) = (1-\gamma^2)(1+\gamma^2) = 1-\gamma^4$

$= 0$

The complex of Mackey functors is

$$
\begin{tikzcd}
\mathbb{Z} \arrow[d, shift right, "1"'] & \mathbb{Z} \arrow[l, "2"'] \arrow[d, shift right, "1+\gamma"'] & \mathbb{Z} \arrow[l, "0"'] \arrow[d, shift right, "1+\gamma"'] & \mathbb{Z} \arrow[l, "2"'] \arrow[d, shift right, "1+\gamma"'] \\
\mathbb{Z} \arrow[u, shift right, "2"'] \arrow[d, shift right, "1"'] & \mathbb{Z}G/H \arrow[l, "\nabla"'] \arrow[u, shift right, "\nabla"'] \arrow[d, shift right, "1"'] & \mathbb{Z}G/H \arrow[l, "2(1-\gamma)"'] \arrow[u, shift right, "\nabla"'] \arrow[d, shift right, "1+\gamma^2"'] & \mathbb{Z}G/H \arrow[l, "1+\gamma"'] \arrow[u, shift right, "\nabla"'] \arrow[d, shift right, "1+\gamma^2"'] \\
\mathbb{Z} \arrow[u, shift right, "2"'] & \mathbb{Z}G/H \arrow[l, "\nabla"'] \arrow[u, shift right, "2"'] & \mathbb{Z}G \arrow[l, "1-\gamma"'] \arrow[u, shift right, "\nabla"'] & \mathbb{Z}G \arrow[l, "1+\gamma"'] \arrow[u, shift right, "\nabla"']
\end{tikzcd}
$$

1 2 3 4

$$
\begin{tikzcd}
\mathbb{Z}/2 \arrow[d, shift right] & 0 \arrow[d, shift right] & \mathbb{Z}/2 \arrow[d, shift right] & 0 \arrow[d, shift right] \\
0 \arrow[u, shift right] \arrow[d, shift right] & \mathbb{Z}/2 \arrow[u, shift right] \arrow[d, shift right] & 0 \arrow[u, shift right] \arrow[d, shift right] & \mathbb{Z}_{+} \arrow[u, shift right] \arrow[d, shift right, "1"'] \\
0 \arrow[u, shift right] & 0 \arrow[u, shift right] & 0 \arrow[u, shift right] & \mathbb{Z}_{-} \arrow[u, shift right, "2"']
\end{tikzcd}
$$

In $\mathbb{Z}G/H$, $(1-\gamma)(1+\gamma^2) = 2(1-\gamma)$

This rep is not oriented.