Note Title 9/14/2012

Equivariant spectra

Recall $\underline{\pi}_* X(G/H) = \pi_*(X^H)$ WHAT IS IT?

but $\underline{H}_* X(G/H) \neq H_*(X^H)$

Some classical homotopy theory.

Freudenthal suspension theorem ~1937

The hom $\pi_{n+k} S^n \longrightarrow \pi_{n+k+1} S^{n+1}$

$S^{n+k} \to S^n$ $\xrightarrow[\text{cone}]{\text{double}}$ $\Sigma S^{n+k} \longrightarrow \Sigma S^n$

$\Sigma S^{n+k} = S^{n+k+1}$, $\Sigma S^n = S^{n+1}$

is onto if $k < n$ and

an isomorphism if $k < n-1$

Same holds if we replace S^n by any $(n-1)$-connected space X.

so $\pi_{k+n} S^n$ is independent of n if $n > k+1$

Want to study very highly connected space

Def A *prespectrum* E is a collection of spaces E_n and maps $\Sigma E_n \to E_{n+1}$. E is a spectrum if the adjoint map $E_n \longrightarrow \Omega E_{n+1}$ = loop space of E_{n+1} is a homeomorphism.

We can convert any prespectrum to

a spectrum by replacing E_n with $\tilde{E}_n = \lim\limits_{k\to\infty} \Omega^k E_{n+k}$

$$\Sigma^k E_n \longrightarrow E_{n+k} \quad \text{iterated structure map}$$

$$E_n \longrightarrow \Omega^k E_{n+k}$$

$$E_n \to \Omega E_{n+1} \to \Omega^2 E_{n+2} \to \Omega^3 E_{n+3} \to \cdots$$

Then by construction $\tilde{E}_n \simeq \Omega \tilde{E}_{n+1}$

We define $\pi_i E = \pi_{n+i} \tilde{E}_n = \pi_{n+i+1} \tilde{E}_{n+1}$

This is independent of n and defined for all integers i.

Examples

1) X = space

$E_n = \Sigma^n X$

$E_{n+1} = \Sigma^{n+1} X = \Sigma E_n$

so the structure map $\Sigma E_n \to E_{n+1}$ is the identity map. The resulting spectrum is the suspension spectrum of X, $\Sigma^\infty X$ or simply X.

e.g. $X = S^0$ so $E_n = S^n$

$\pi_k E = \pi_{n+k} S^n$ for $n \gg 0$.

2) Let A be an abelian gp. There is an Eilenberg - MacLane space $K(A,n)$ with

$$\pi_i K(A,n) = \begin{cases} A & \text{if } i = n \\ 0 & \text{if } i \neq n \end{cases}$$

Thm says $H^n(X;A) = [X, K(A,n)]$

There is a htpy equivalence

$$K(A,n) \simeq \Omega K(A,n+1)$$

$$\Sigma K(A,n) \longrightarrow K(A,n+1)$$

We can define a pre-spectrum E by $E_n = K(A,n)$ with the structure maps above.

This is the Eilenberg-MacLane spectrum HA.

<u>Thm</u> For a space or spectrum X

$$H_n(X;A) = \pi_n(X \wedge HA).$$

Fancier definition.

We want a space $E(V)$ for each finite dimensional Euclidean vector space V. Each V is a subspace of an infinite dimensional Euclidean vector space $\mathcal{U}$ called a <u>universe</u>. The htpy type of $E(V)$ depends only on $\dim(V)$.

Given orthogonal subspaces V, W of $\mathcal{U}$, we get a structure map

$$S^W \wedge E(V) \longrightarrow E(V+W)$$

where $S^W = W \cup \{\infty\}$.

More precise description

Let $\mathcal{I}$ be the category whose objects are fin dim $V \subset \mathcal{U}$ and whose morphisms are orthogonal maps. E is a functor from $\mathcal{I}$ to Spaces with some additional condition for structure maps.

This is an <u>orthogonal spectrum</u>.

<u>Going equivariant</u>

V = vector space with orth G-action

$E(V)$ = G-space for each V.

$\mathcal{I}(G)$ = category of finite dim

G-invariant subspace $\mathcal{U}$.

$\mathcal{U}$ is a complete universe if it contains every fin dimensional orth rep of G.

An orthogonal G-spectrum is a function $I(G) \longrightarrow$ G-spaces.

$$S^V \wedge E(W) \longrightarrow E(V+W)$$