MATH 550

Note Title 9/17/2012

Patrick's talk on Mackey functors

Greenlees - May 1992

Goal: Find a simple set of Mackey functors from which all can be constructed.

G = finite abelian gp.

Notation: $G\text{Set}$ = category of finite G-sets

$\mathcal{A}b$ = " abelian gps.

Recall A Mackey functor $\underline{M}$ is contravariant / covariant functor

$$G\text{Set} \longrightarrow \mathcal{A}b$$

1) $\amalg \longmapsto \oplus$

2) For

$$\begin{array}{ccc} P & \xrightarrow{\delta} & S \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\alpha} \\ R & \xrightarrow{\beta} & T \end{array}$$

pullback diagram

$$\alpha^* \beta_* = \delta_* \gamma^* : \underline{M}(R) \longrightarrow \underline{M}(T).$$

<u>Def</u> A contravariant functor additive on disjoint unions is called a <u>Bredon coefficient system</u>.

Let $M[G]$ be the category of Mackey functors

G-mod = category of $\mathbb{Z}[G]$-modules

We define two functors

$$M[G] \underset{R}{\overset{L}{\rightleftarrows}} G\text{-mod} \qquad L\underline{M} := \underline{M}(G/e)$$

$$R\,V(G/H) := V^H$$

fixed pt Mackey functor

<u>Prop</u> For a G-module V and Mackey functor $\underline{M}$, L and R are left + right adjoint

i.e. $\mathrm{Hom}_{G\text{mod}}(L\underline{M}, V) \cong \mathrm{Hom}_{M[G]}(\underline{M}, RV)$

Let K be a (normal) subgp of G

$$G \xrightarrow{\varepsilon} G/K \qquad \text{quotient gp}$$

Define a category $\mathcal{M}[G]/K$ whose objects are Mackey functors $\underline{M}$ with $\underline{M}(G/H) = 0$ unless $K \subseteq H$. This is the <u>category of Mackey functors concentrated over K</u>.

Let $G\mathcal{S}et^K$ = category of G-sets fixed by K.

pullback along ε

$$\begin{tikzcd}
G/K\text{-sets} \arrow[d, shift right, "\varepsilon^{\#}"'] \\
G\mathcal{S}et^K \arrow[u, shift right, "\varepsilon_{\#}"']
\end{tikzcd} \qquad \text{isomorphism}$$

Let $\underline{M}' \in \mathcal{M}[G/K]$

$$G\mathcal{S}et^K \xrightarrow{\varepsilon_{\#}} G/K\,\mathcal{S}ets \xrightarrow{\underline{M}'} \mathcal{A}b$$

The composite is a Mackey functor concentrated over K.

Consider the diagram

$$\begin{tikzcd} G/H \text{ mod} \arrow[d, "R"] & \\ M[G/H] \arrow[r, "\varepsilon_*"] & M[G]/H \end{tikzcd}$$

Note 1. $RV(G/J) = 0$ if $H \nsubseteq J$.

2. $RV(G/H) = V$ for a G/H-module V.

Main Thm 12 Let $\mathcal{A}$ be a collection of G-Mackey functors s.t. 1) RV is in $\mathcal{A}$ for all $H \subseteq G$ and G/H-modules V.

2) If $0 \to N' \to N \to N'' \to 0$ is SES with any 2 in $\mathcal{A}$, so is the

third. (Closure). Then $\mathcal{A}$ is the collection of all Mackey functors.

Pf (For non-abelian G, use NH/H-module V.)

Partition the subgps of G

$S_0 = \{e\}$. S_j = collection of subgps not in S_{j-1} but each proper subgp is in some S_i for $i<j$.

e.g. $G = C_2 \times C_2 = \{e, (0,1), (1,0), (0,1)\}$

$S_0 = \{e\}$

$S_1 =$ set of subgps of order 2

$S_2 = \{G\}$

For some m, $S_m = \{G\}$.

Choose a subgp H_j from each S_j

Let $S_j = \{H_{j1}, H_{j2}, \cdots\}$

$S_m = \{G\}$

Choose $H_0 = e$

$H_1 = H_{1,k_1}$

$H_2 = H_{2,k_2}$

$\vdots$

<u>Def</u> A Mackey functor $\underline{M}$ is of type j if $\underline{M}(G/H_j) \neq 0$ but $M(G/H_{j'}) = 0$ for $j' < j$.

<u>Proceed by induction</u>

Suppose $\underline{M}$ has type j. Assume all type j' functors for j' are in A.