Patrick's talk on Mackey functors

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Goal: Find a simple set of Mackey functors from which all can be constructed.

$G = \text{finite abelian gp.}$

Notation: $\text{C-set} = \text{category of finite } G \text{-sets}$

$\text{Ab} = \text{category of abelian gqs.}$

Recall a Mackey functor $M$ is a contravariant/covariant function

$$G\text{-set} \rightarrow \text{Ab}$$

1) $\text{Triv}$

2) For

$$\begin{array}{ccc}
R & \xrightarrow{p} & S \\
\downarrow & & \downarrow \\
B & \xrightarrow{f} & R
\end{array}
$$

full back diagram
\[ \mathbb{Z}^* \beta^* = \delta^* \delta^* : M(R) \rightarrow M(T). \]

**Def.** A contravariant functor additive on disjoint unions is called a Braided coefficient system.

Let \( \mathcal{M}[G] \) be the category of Mackey functions.

\( C \text{-} \text{mod} = \text{category of } Z[G] \text{-} \text{modules} \)

We define two functors:

\[ \mathcal{M}[G] \xrightarrow{L} C \text{-} \text{mod} \]

\[ \xleftarrow{R} \]

\[ L M := M(G/e) \]

\[ R V(C/H) := V^H \]

**Prop.** For a \( C \)-module \( V \) and Mackey functor \( M \), \( L \) and \( R \) are left and right adjoint, i.e. \( \text{Hom}_{C \text{-} \text{mod}}(LM, V) \cong \text{Hom}_{\mathcal{M}[G]}(M, RV) \)
Let \( K \) be a (normal) subset of \( G \).
\[ G \to G/K \text{ quotient map} \]
Define a category \( M[G, K] \) whose objects are Mackey functions \( M \) with \( M(G/H) = 0 \) unless \( K \subseteq H \). This is the category of Mackey functions concentrated over \( K \).

Let \( C \text{-}\mathbf{Set}^K = \text{category of } \mathbf{C}\text{-sets fixed by } K \).

\[
\begin{array}{ccc}
C/K & \xrightarrow{\mathcal{E}_#} & C/\mathbf{Set}^K \\
& \searrow & \downarrow \text{restriction} \\
& \mathcal{E}_# & \\
& & \\
& & \\
\end{array}
\]

Let \( M' \in M[G, K] \)
\[ C/\mathbf{Set}^K \xrightarrow{\mathcal{E}_#} C/K \text{ sets} \xrightarrow{M'} \text{ Alg} \]
The composite is a Mackey functor concentrated over $K$.

Consider the diagram

$$
\begin{array}{c}
G/H \xrightarrow{\pi} \text{M}(G/H) \\
\downarrow R \\
\text{M}(G/H) \xrightarrow{\phi} \text{M}(G)/H
\end{array}
$$

Note 1. $RV(G/H) = 0$ if $H \neq J$.

1. $RV(G/H) = V$ for a $G/H$-module $V$.

Main Claim: Let $A$ be a collection of Mackey functors

1) $RV$ is in $A$ for all $H \leq G$ and $G/H$-modules $V$.

2) If $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$ is any SES with any $Z$ in $A$, $\phi$ is the
third. (Closure). Then A is the collection of all Mackey functors.

Proof. (For non-abelian G, use NH/H-module.)

Partition the subgroups of G:

\[ S_0 = \{ e \} \quad S_j : \text{collection of subgroups not in } S_{j-1} \text{ but each proper subgroup in some } S_i \text{ for } i < j \]

E.g. \[ G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{ e, (0,1), (1,0), (0,1) \} \]

\[ S_0 = \{ e \} \]

\[ S_1 = \text{set of subgroups of order } 2 \]

\[ S_2 = \{ G \} \]

For some \( m \), \[ S_m = \{ G \} \]

Choose a subgroup \( H \) from each \( S_j \).
Let \( S_j = \sum H_{01}, H_{02}, \ldots, H_j \)
\[ S_m = \sum C_j \]

Choose \( H_0 = 0 \)
\( H_1 = H_{11}, k_1 \)
\( H_2 = H_{21}, k_2 \)

A Mackey functor \( M \) is of type \( j \) if
\[ M(G/H_i) \neq 0 \text{ but } M(G/H_{i'}, 0 \text{ for } j < j'}. \]

Proced by induction
Support \( M \) has type \( j \). Assume all type \( j \) functions for \( j < j \) and
in \( A \).