More of Patrick's talk

Thm 2. Let $A$ be a collection of Mackey functions such that

1) $\text{RVE} A$ for all $H,G_i$ and all $G/H$-modules $V$

2) If $0 \to N' \to N \to N'' \to 0$ is a SES

where two are in $A$, the third
in $\text{RVE} A$

Then $A$ is the collection of all Mackey functions

Proof: Partition the subgroups of $G_i$ as follows

$S_0 = \{ e \}$

$S_i = \{ \text{subgps not in } S_{i-1} \}$, where each

broken subgroup is in $S_i$, for $i \leq j$.

Since $G_i$ is finite, $S_m = \{ e \}$ for some $m$. 

Order and list the analogs as follows

$S_0 = \{ e \}$

$S_1 = \{ H_{i_1}, H_{i_2}, \ldots, H_{i_m} \}$

$S_0 = \{ H_{i_1}, H_{i_2}, \ldots, H_{i_m} \}$

$S_m = \{ H_{i_1}, H_{i_2}, \ldots, H_{i_m} \}$

$S_m = \{ H_{i_1}, H_{i_2}, \ldots, H_{i_m} \}$

Def: A Mackey functor $M$ is of type $(i, k)$ if $M(C/H, i) \neq 0$, and

$M(C/H, i) = 0$ for all $j \neq i$ and $y' = y, k < k'$

Proceed by downward induction.

Note: If $M(C/H)$ for all broken $H$, then

$M = RM(C/H)$

Assume $M$ is of type $(i, k)$ and all
If greater than in A.

meaning tight \((j,k)\) with \(j > j'\) or \(j = j'\) and \(k > k'\).

There is a map \(N : M \rightarrow RM(G/H_{i,i},k)\).

Consider the SES

1) \(0 \rightarrow \ker N \rightarrow M \rightarrow \text{im } N \rightarrow 0\)

2) \(0 \rightarrow \text{im } N \rightarrow \text{RM}(G/H_{i,i},k) \rightarrow \text{coker } N \rightarrow 0\)

If \(\text{coker } N \in A\), then \(\text{im } N \in A\).

If \(\text{im } N\), \(\ker N \not\in A\), then \(M \not\in A\).

SUFFICE to show \(\ker N\), \(\text{coker } N \in A\).

By construction both \(\ker N\) and \(\text{coker } N\)

vanish on \(G/H_{i,i}\), and are in \(A\) by induction. QED
Properties of $G$-spectra

See HHR Appendix A for details of the definitions.

**Examples** Let $V$ be a representation of $G$. Define a $G$-spectrum $E = \sum_{n} S^{V}$ (by abuse of notation). For all $W$, $E_{W} = S^{V} + W$. Define $E = S^{-V}$ by $E_{W} = S^{W-V}$ for all $W \supset V$, where $W-V$ is orthogonal complement of $V$ in $W$.

**NOTE:** To get a spectrum $E$, it suffices to define $E_{W}$ for a cofinal collection of $W$'s.
e.g. for all W containing V, we can define $S_{W-V} = S_{W} \cap S_{V}$, as previously defined.

We get a sphere spectrum for each virtual rep. $W-V$.

Let $\Pi_{W-V} X = \left[ S^W, X \right] G$

$= \left[ S^0 \vee S^{V-W-V} X \right] G$

$= \Pi_0 (G(V-W-V)X)$

We get homotopy gfs graded by RO($G$), the orthogonal rep ring of $G$.

Collectively, these gfs are denoted $\Pi_{W-V} X \vee \pi_{-W-V}$.
\[ \text{Let } X = \text{ 2-graded homotopy gfs of } X. \]

Quick remark in \( RO(G) \): \([G \text{ finite}] \)

Let \( R(G) = \text{ complex rep ring of } G \).

Reference: Serre's book

Linear reps of finite gfs.

Let \( V \) be a complex rep of \( G \).

(finite dimensional). Choose a basis for \( V \), so we have a hom \( G \to \text{GL}_n(\mathbb{C}) \)

\( \text{so we have a hom } \)

\( G \to \text{GL}_n(\mathbb{C}) \)

For each \( g \in G \), consider \( \text{Tr}_n(\rho(g)) \)

\( = \text{trace } = \text{sum of diagonal entries.} \)
3. **trace**

1) Trace is independent of basis choice

2) If \( g \) and \( g' \) are conjugate, then they have the same trace

3) We have a map \( \mathbb{R}(G) \rightarrow \text{ring Cl}(G) \) of completed valued fields in the set of conjugacy classes in \( G \).

It is a ring homomorphism, and \( \mathbb{R}(G) \otimes_{\mathbb{Z}} \mathbb{C} = \text{Cl}(G) \).

But the rank of \( \mathbb{R}(G) \) as a free abelian group is the number of conjugacy classes. The number of irreducible \( \mathbb{C} \)-valued irreducible representations is the number of conjugacy classes.
classes in character tables.

$C_4$ has 4 columns, one for each conjugacy class. There are 4 rows for each irreducible rep.

$C_4 = C_4$ with generator $y$

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<th>$1$</th>
<th>$y$</th>
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