MATH 550

Note Title 9/19/2012

More of Patrick's talk

Thm 12 Let $\mathcal{A}$ be a collection of Mackey functors such that

1) $R_V \in \mathcal{A}$ for all $H \subseteq G$ and all G/H-modules V

2) If $0 \to N' \to N \to N'' \to 0$ is a SES where two are in $\mathcal{A}$, the third is also

Then $\mathcal{A}$ is the collection of all Mackey functors

Proof: Partition the subgps of G as follows

$S_0 = \{e\}$

$S_j =$ subgps not in S_{j-1} where each proper subgp is in S_i for $i < j$.

Since G is finite, $S_n = \{G\}$ for some n.

Order and list the subgps as follows

$S_0 = \{e\}$

$S_1 = \{H_{1,1}, H_{1,2} \cdots H_{1,n_1}\}$

$\vdots$

$S_j = \{H_{j,1}, H_{j,2}, \cdots H_{j,n_j}\}$

$\vdots$

$S_m = \{H_{m,1}\} = \{G\}$

Def A Mackey functor $\underline{M}$ is of type (j,k) if $M(G/H_{j,k}) \neq 0$ but $M(G/H_{j',k'}) = 0$ for all $j' < j$ and $j' = j$, $k' < k$.

Proceed by downward induction.

Note if $M(G/H)$ for all proper H, then

$\underline{M} = R\,M(G/G)$

Assume M is of type (j,k) and all

$\underline{M}$ of greater type are in $\mathcal{A}$.
(meaning type (j',k') with $j' > j$ or $j' = j$ and $k' > k$)

There is a map $\eta : \underline{M} \longrightarrow RM(G/H_{j,k})$

Consider the SES

1) $0 \to \ker \eta \longrightarrow \underline{M} \longrightarrow \operatorname{im} \eta \longrightarrow 0$

2) $0 \to \operatorname{im} \eta \longrightarrow R\underline{M}(G/H_{j,k}) \longrightarrow \operatorname{coker} \eta \longrightarrow 0$

$\cancel{\in \mathcal{A}}$

If $\operatorname{coker} \eta \in \mathcal{A}$, then $\operatorname{im} \eta \in \mathcal{A}$

If $\operatorname{im} \eta$, $\ker \eta \in \mathcal{A}$ then $\underline{M} \in \mathcal{A}$.

Suffices to show $\ker \eta$, $\operatorname{coker} \eta \in \mathcal{A}$.

By construction both $\ker \eta$ and $\operatorname{coker} \eta$ vanish on $G/H_{j,k}$ and are in $\mathcal{A}$ by induction. QED

Properties of G-spectra

See HHR appendix A for details of the definitions.

Examples Let V be a representation of G. Define a G-spectrum $E = \Sigma^\infty S^V$ (or S^V by abuse of notation) by $E_W = S^{V+W}$ for all W.

Define $E = S^{-V}$ by

$$E_W = S^{W-V} \quad \text{for all } W \supset V$$

where $W-V$ = orthogonal complement of V in W.

NOTE: To get a G-spectrum E, it suffices to define E_W for a cofinal collection of W's e.g. for all W's containing V.

Can define $S^{W-V} = S^W \wedge S^{-V}$

as previously define.

We get a sphere spectrum for each virtual rep. $W-V$.

Let $\pi_{W-V} X = [S^{W-V}, X]^G$

$$= [S^0, S^{V-W} \wedge X]^G$$

$$= \pi_0 (S^{V-W} \wedge X)$$

We get homotopy gps graded by $RO(G)$, the orthogonal rep ring of G.

Collectively these gps are denoted by $\pi_\star X$ \pi_\star

$\pi_* X$ = $\mathbb{Z}$-graded homotopy gps of X.

<u>Quick course in $RO(G)$:</u> [G finite]

Let $R(G)$ = complex rep ring of G

reference: Serre's book
Linear reps of finite gps.

Let V be a complex rep of G (finite dimensional). Choose a basis for V, so we have a hom

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[dr] & GL_n(\mathbb{C}) \\ & U(n) \arrow[u] \end{tikzcd}$$

For each $\gamma \in G$ consider $Tr(\rho(\gamma))$
= trace = sum of diagonal entries.

3 miracles:

1) Trace is independent of basis choice
2) If γ and γ' are conjugate, then they have the same trace
3) We have a map

$R(G) \longrightarrow$ ring $Cl(G)$ of complex valued fns in the set of conjugacy classes in G.

It is a ring hom and

$$R(G) \otimes_{\mathbb{Z}} \mathbb{C} = Cl(G)$$

Cor The rank of $R(G)$ as a free abelian gp is the # of conjugacy classes. The # of irreducible reps is the # of conjugacy classes k.

Character tables.

k rows + columns

column for each conj class

row for each irred rep.

$G = C_4$ with generator γ

		1	γ	γ^2	γ^3
	1	1	1	1	1
$\gamma \mapsto [i]$	λ	1	i	-1	$-i$
$\gamma \mapsto [-1]$	λ^2	1	-1	1	-1
$\gamma \mapsto [-i]$	λ^3	1	$-i$	-1	i