Note Title 9/21/2012

Recall the character table for $C(4)$

	1	γ	γ^2	γ^3
1	1	1	1	1
λ	1	i	-1	$-i$
λ^2	1	-1	1	-1
λ^3	1	$-i$	-1	i
$\lambda+\lambda^3$	2	0	-2	0

$\bar{\lambda} = \lambda^3$

$\gamma \mapsto [-1]$

$\gamma \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ rotation by $\pi/2$

Thm If the character of a rep W over $\mathbb{C}$ is real, then $W = \mathbb{C} \otimes_{\mathbb{R}} V$

Let σ denote the sign rep $\gamma \mapsto [-1]$

There is a real rep V with $\mathbb{C} \otimes_{\mathbb{R}} V = \lambda + \lambda^3$

Call this rotation λ.

Real character table for C_4

$$\begin{array}{c|cccc} & 1 & \gamma & \gamma^2 & \gamma^3 \\ \hline 1 & 1 & 1 & 1 & 1 \\ \sigma & 1 & -1 & 1 & -1 \\ \lambda & 2 & 0 & -2 & 0 \end{array}$$

Under λ

$\gamma \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\gamma^2 \mapsto \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$RO(C_4) = \mathbb{Z}\{1, \sigma, \lambda\}$ $\quad \rho_4 = 1 + \sigma + \lambda$

$R(C_4) = \mathbb{Z}\{1, \lambda, \lambda^2, \lambda^3\}$

There is a map $\quad RO(G) \longrightarrow R(G)$

$V \longmapsto \mathbb{C} \otimes_{\mathbb{R}} V$

Its image is $\quad R(G)^{Gal(\mathbb{C}:\mathbb{R})}$

We computed $\underline{H}_*(S^{\rho_4})$.

Classical fact: For a space X,

$$H_* X = \pi_*(X \wedge H\mathbb{Z})$$

$H\mathbb{Z}$ = Eilenberg–MacLane spectrum.

where $\pi_i H\mathbb{Z} = \begin{cases} \mathbb{Z} & \text{if } i=0 \\ 0 & \text{if } i \neq 0 \end{cases}$

There is an equivariant analog

There is a spectrum $H\underline{\mathbb{Z}}$ with

(1) $$\underline{\pi}_i(H\underline{\mathbb{Z}}) = \begin{cases} \underline{\mathbb{Z}} & \text{if } i=0 \\ 0 & \text{if } i \neq 0 \end{cases} \quad i \in \mathbb{Z}$$

Here $\underline{\mathbb{Z}}$ is the constant $\mathbb{Z}$-valued Mackey functor in which each restriction map is the identity

For $G = C_4$ the diagram is

$$\begin{tikzcd} \underline{\mathbb{Z}} \arrow[d, bend right, "1"'] \\ \underline{\mathbb{Z}} \arrow[u, bend right, "2"'] \arrow[d, bend right, "1"'] \\ \underline{\mathbb{Z}} \arrow[u, bend right, "2"'] \end{tikzcd}$$

For a G-space (or spectrum) X

$$\underline{H}_* X = \underline{\pi}_* (X \wedge H\underline{\mathbb{Z}})$$

$i - \rho_4 \in RO(C_4)$

$$\underline{H}_i S^{\rho_4} = \underline{\pi}_i (S^{\rho_4} \wedge H\underline{\mathbb{Z}}) = \underline{\pi}_{i-\rho_4} H\underline{\mathbb{Z}}$$

$$\underline{\pi}_i \, X(G/H) = \pi_i (X^H)$$

$G = C_4$

$$\underline{\pi}_i (S^{\rho_4} \wedge H\underline{\mathbb{Z}})(G/H) = \pi_i ((S^{\rho_4} \wedge H\underline{\mathbb{Z}})^H)$$

WARNING. Fixed points do NOT

respect smash products.

e.g. $(S^{\rho_4} \wedge H\underline{\mathbb{Z}})^H \neq (S^{\rho_4})^H \wedge (H\underline{\mathbb{Z}})^H$

recall $(S^{\rho_4})^H = \begin{cases} S^1 & H = G \\ S^2 & H = C_2 \\ S^4 & H = e \end{cases}$

$\pi_i\, H\underline{\mathbb{Z}}\,(G/H) = \pi_i\left(H\underline{\mathbb{Z}}^H\right) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$

so $(H\underline{\mathbb{Z}})^H = H\mathbb{Z}$ for each H

the gp action is trivial.

$\pi_i\left((S^{\rho_4})^H \wedge (H\underline{\mathbb{Z}})^H\right) = H_*\,(S^{\rho_4})^H$ = known

$\pi_i\left((S^{\rho_4} \wedge H\underline{\mathbb{Z}})^H\right)$ = different answer

Fixed points do not like smash products.

More weirdness. Let S^0 have trivial G-action

Thm (tom Dieck 1979)

$\underline{\pi}_0(\Sigma^\infty S^0) = \underline{A}$ = Burnside ring Mackey functor

(not $\underline{\mathbb{Z}}$!)

Recall $A(G)$ = gp completion of the monoid of iso classes of finite G-sets.

$= \mathbb{Z}\{G/H : H \text{ up to conjugacy}\}$

For a G-set S,

$\underline{A}(S)$ = gp completion of the monoid of iso classes of finite G-sets over S, i.e. maps $T \to S$.

e.g. $\underline{A}(G/H) = \mathbb{Z}\{T \to G/H\} = A(H)$

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