MATH 550

Note Title 9/5/2012

Reading for the next week or so
Adams §1–2
Greenlees–May 1995 §1
Greenlees–May 1992 – some student to talk on it ~9/14 assuming G is abelian

Def Let X and Y be G-spaces. A cont map $X \xrightarrow{f} Y$ is equivariant if (f is a G-map)
$f(\gamma(x)) = \gamma(f(x))$ for each $\gamma \in G$.

Alternate formulation in terms of mapping spaces: Let Map(X, Y)

be the space of <u>all</u> cont maps $X \to Y$ (use compact open topology). G acts on $\mathrm{Map}(X,Y)$ as follows. Let $\gamma \in G$

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[rrr, bend right, "\gamma(f)"'] & X \arrow[r, "f"] & Y \arrow[r, "\gamma^{-1}"] & Y \end{tikzcd}$$

This makes $\mathrm{Map}(X,Y)$ a G-space.

$\mathrm{Map}(X,Y)^G$ = fixed set
= space of equivariant maps

$\gamma(f) = f$ means for each $\gamma \in G$ this diagram commutes

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ X \arrow[u, "\gamma"] \arrow[r, "f"] & Y \arrow[u, "\gamma"'] \end{tikzcd}$$

i.e. $\gamma(f(x)) = f(\gamma(x))$.

__Def__ Two G-maps f_0, f_1 are __G-homotopic__ if there is a G-map

$$I \times X \xrightarrow{h} Y$$

s.t. $h(0,x) = f_0(x)$ and $h(1,x) = f_1(x)$.

A G-map $X \xrightarrow{f} Y$ is a __G-equivalence__ if $\exists$ a G-map $Y \xrightarrow{g} X$ s.t.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right] & Y \arrow[r, "g"] \arrow[rr, bend left] & X \arrow[r, "f"] & Y \end{tikzcd}$$

$$gf \underset{G}{\simeq} 1_X \quad \text{and} \quad fg \underset{G}{\simeq} 1_Y.$$

Recall for a finite gp G

$$\left\{\begin{array}{l}\text{Isomorphism}\\ \text{classes of}\\ \text{finite } G\text{-sets}\end{array}\right\} \longleftrightarrow \begin{array}{l}\text{Free abelian monoid}\\ \text{on the sets } G/H\\ \text{for conjugacy classes}\\ \text{of subgps } H \subset G.\end{array}$$

Remarks: This set of iso classes has a multiplication induced by Cartesian product of sets.

This can be converted to a ring by considering virtual G sets $X - Y$.

This is the Burnside ring of G $A(G)$.

Recall the definition of a CW-complex X.

$S_0 = X^0 =$ discrete set

X^n is obtained from X_{n-1} by
attaching a set S_n of cells
D^n = n-disk via attaching maps

$$S_n \times \partial D^n \xrightarrow{f_n} X^{n-1}$$

$\shortparallel$

disjoint union
of $(n-1)$-spheres

X^n = mapping cone of f.

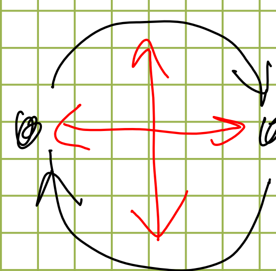
We require each S_n to be a G-set
and each f_n a G-map. This is a G-CW complex.

Thm A G-map $X \xrightarrow{f} Y$ of G-CW complexes
is a G-equivalence iff for each
subgp $H \subset G$ $\quad X^H \xrightarrow{f^H} Y^H$ is an

ordinary htpy equivalence.

Example ① $G = C_2$, $X = S^n$ with antipodal action. $X^0 = G = S_0$

We will set $S_i = G$ for $0 \le i \le n$

$X^1 =$ 

$X^2 = X^1 \cup (G \times D^2)$

$\lim_{n \to \infty} S^n = S^\infty =$ contractible free G-space.

Example Construction of contractible free G-space. The join of 2-spaces X, Y, $X * Y = X \times [0,1] \times Y / \sim$

$(x, 0, y) \sim (x, 0, y')$

$(x, 1, y) \sim (x', 0, y)$

exercise: $S^m * S^n = S^{m+n+1}$

If X is $(m-1)$-connected and Y is $(n-1)$-connected then $X*Y$ is $(m+n)$-connected.

If X and Y are (free) G-spaces then so is $X*Y$. G acts trivially on $[0,1]$

Let $E_n G = \underbrace{G * G * \cdots * G}_{n+1 \text{ factors}}$

This is $(n-1)$ connected and has free G-action

For $G = C_2$ this is S^n as above.

There are inclusion maps $E_n G \to E_{n+1} G$

Let $EG = \bigcup E_n G$. It is a

contractible free G-space.

Each $E_n G$ is a G-CW complex.