MATH 550

Note Title 9/7/2012

Recall G-CW complexes.

For an ordinary CW cx X we have a cellular chain cx $C(X)$ defined

$C_n(X)$ = free abelian gp on n-cells
= free abelian gp on the set S_n.

In equiv case, S_n is a G-set, making $C_n(X)$ a $\mathbb{Z}[G]$, the gp ring of G.

<u>Def</u> The <u>gp ring</u> $\mathbb{Z}[G]$ of a gp G is the free abelian gp on the set G. For $\gamma \in G$, let $[\gamma]$ be the corresponding generator of

$1 = [e]$ this abelian gp. The multiplication in $\mathbb{Z}[G]$ is given by $[\gamma_1][\gamma_2] = [\gamma_1\gamma_2]$.

For a G-set S, the free abelian gp generated by S is a $\mathbb{Z}[G]$-module. Such a $\mathbb{Z}[G]$-module is called a <u>permutation module</u>.

<u>Example</u> $G = C_n$ with generator γ.

$$\mathbb{Z}[G] = \mathbb{Z}\left[[\gamma]\right] / \left([\gamma]^n - 1\right)$$

For a subgp $H \subset G$, let G/H = set of left cosets = G-set

$\mathbb{Z}[G/H]$ = free ab gp on G/H

= permutation module

For $H \subset K \subset G$

$G/H \longrightarrow G/K$ map of G-sets

$$\mathbb{Z}[G/H] \underset{\Delta}{\overset{\nabla}{\rightleftarrows}} \mathbb{Z}[G/K]$$

∇ = \nabla

Δ = \Delta

Let $\{\gamma_1, \gamma_2 \cdots\}$ be a set of coset rep's for K/H. Given $x \in G/K$, we have $\sum [\gamma_i] x \in G/H$.

e.g. $H = e$ and $K = G$.

$\mathbb{Z}[G/K] = \mathbb{Z}$ with trivial G-action

$$\mathbb{Z}[G/H] \cong \mathbb{Z}[G]$$

The diagonal map $\quad \mathbb{Z} \xrightarrow{\Delta} \mathbb{Z}[G]$

assume G is finite. $\quad 1 \longmapsto \sum_{\gamma \in G} [\gamma]$

Exercise: Find the endomorphisms $\Delta\nabla$ and $\nabla\Delta$ above.

$H_*(C(X))$ is a graded $\mathbb{Z}[G]$-module.

More about this later.

Homotopy gps: We need a base point that is fixed by G.

Given a G-space X, let X_+ be the disjoint union of X with a base point fixed by G.

The nth htpy gp of a pointed G-space $\underline{X}$ is a functor (finite G-sets S) to (abelian gps)

$$S \longmapsto [S_+ \wedge S^n, \underline{X}]^G_*$$

= htpy classes of equiv base pt preserving maps.

Call this $\underline{\pi}_n^G(\underline{X})$

Formal properties of this functor
1) Contravariant
2) Converts disjoint unions to direct sums.

The functor $\underline{\pi}_n^G(X)$ is determined by its values on the G-sets G/H for conjugacy classes of subgps $H \subset G$.

For $H \subset K \subset G$ we have a map of G-sets $G/H \longrightarrow G/K$, giving

$$\underline{\pi}_n X(G/H) \xleftarrow{\mathrm{Res}_H^K} \underline{\pi}_n X(G/K)$$

This is called a restriction map.

Def A Mackey functor $\underline{M}$ is a functor

finite G-sets $\longrightarrow$ abelian gps

1) Contravariant
2) Additive on disjoint unions.

In addition to the restriction maps above we have

$$\underline{M}(G/H) \underset{\mathrm{Ind}_H^K}{\overset{\mathrm{Res}_H^K}{\rightleftarrows}} \underline{M}(G/K)$$

This is called a transfer map.

Example ① $\underline{M}(G/H) = \mathbb{Z}[G/H]$
with $\mathrm{Res}^K_H = \nabla$, $\mathrm{Tr}^K_H = \Delta$

② Let M be a $\mathbb{Z}[G]$-module,
i.e. an abelian gp equipped with
a G-action. Define a Mackey
functor $\underline{M}$ by

$$\underline{M}(G/H) = M^H$$

Can define transfer explicitly.

Call this a fixed point Mackey functor.