THE SLICE SPECTRAL SEQUENCE OF MACKEY FUNCTORS FOR THE C_4 ANALOG OF REAL K-THEORY

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WORK IN PROGRESS

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The purpose of this paper is to describe the slice spectral sequence of a 32-periodic C_4-spectrum $K_{\mathbf{H}}$ (to be defined in §5) related to the C_4 norm $MU^{((C_4))} = N_2^4 MU_{\mathbf{R}}$ of the real cobordism spectrum $MU_{\mathbf{R}}$. Part of this spectral sequence is illustrated in an unpublished poster produced in late 2008 and shown at the end of this paper. It shows the spectral sequence convering to the homotopy of the fixed point spectrum $K_{\mathbf{H}}^{C_4}$. Here we will describe the corresponding spectral sequence of Mackey functors converging to the graded Mackey functor $\underline{\pi}_* K_{\mathbf{H}}$. The C_8 analog

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of $K_{\mathbf{H}}$ is 256-periodic and detects the Kervaire invariant classes θ_j. The C_2 analog is the real K-theory spectrum $K_{\mathbf{R}}$.

We will rely extensively on the results, methods and terminology of [HHR].

1. General nonsense about equivariant stable homotopy theory

We first discuss some structure on the homotopy groups of a G-spectrum X. For each representation V we get a Mackey functor $\underline{\pi}_V^G X = \underline{\pi}_0^G \Sigma^{-V} X$; we will often suppress G from the notation when it is clear from the context. Its components are the ordinary homotopy groups of various fixed point sets. In [HHR, 2.2.5] the group

$$\underline{\pi}_k^G X(G/H) = \pi_k(X^H)$$

(for an integer k) is denoted by $\pi_k^H X = [S^k, X]^H$. Here S^k has the trivial group action, so an H-equivariant map to X must land in the fixed point spectrum X^H. Thus

$$\pi_k^H X = \pi_k(X^H),$$

the ordinary kth homotopy group of the ordinary spectrum X^H. Since the Weyl group of H acts on X^H, this group is a module over it.

For a representation V of G, the group

$$\underline{\pi}_V^G X(G/H) = \pi_V^H X = [S^V, X]^H$$

is isomorphic to

$$[S^0, S^{-V} \wedge X]^H = \pi_0(S^{-V} \wedge X)^H.$$

However fixed points do not respect smash products, so we cannot equate this group with

$$\pi_0(S^{-V^H} \wedge X^H) = [S^{V^H}, X^H] = \pi_{|V^H|} X^H = \underline{\pi}_{|V^H|}^G X(G/H).$$

Conversely a G-equivariant map $S^V \to X$ represents an element in

$$[S^V, X]^G = \pi_V^G X = \underline{\pi}_V^G X(G/G).$$

For $K \subseteq H \subseteq G$ we have maps

$$\begin{array}{ccc} \underline{\pi}_V^G X(G/H) & \underset{\mathrm{Res}_K^H}{\overset{\mathrm{Tr}_K^H}{\rightleftarrows}} & \underline{\pi}_V^G X(G/K) \\ \| & & \| \\ \pi_0(\Sigma^{-V} X)^H & & \pi_0(\Sigma^{-V} X)^K \end{array}$$

which we call the *fixed point restriction and transfer maps*. When X is a ring spectrum, we have the *fixed point Frobenius relation*

$$\mathrm{Tr}_K^H(\mathrm{Res}_K^H(a) b) = a \mathrm{Tr}_K^H(b) \qquad \text{for } a \in \underline{\pi}_\star X(G/H) \text{ and } b \in \underline{\pi}_\star X(G/K). \tag{1}$$

In particular this means that

$$a \mathrm{Tr}_K^H(b) = 0 \qquad \text{when } \mathrm{Res}_K^H(a) = 0. \tag{2}$$

We can also regard X as an H-spectrum for any subgroup H of G; we will not make a notational distinction between these two structures on X. As such it has an $RO(H)$-graded Mackey functor over H of homotopy groups. For simplicity we assume now that G is abelian. Recall that for a Mackey functor $\underline{M}$ over H, the abelian group $\underline{M}(H/K)$ (for a subgroup K of H) is a module over $\mathbf{Z}[H/K]$. For the $RO(H)$-graded abelian group $\underline{\pi}_\star^H X(H/K)$ for a G-spectrum X, this module structure extends to one over $\mathbf{Z}[G/K]$.

We will define maps relating these Mackey functors over the various subgroups and call them *group action restriction and transfer maps*, denoted by $\underline{r}_K^H$ and $\underline{t}_K^H$. The map $\underline{r}_H^G$ is induced by the forgetful functor from G-spectra to H-spectra denoted in [HHR, 2.2.4] by i_H^*; for trivial H it is denoted by i_0^*.

Given a representation V of G restricting to W on $H \subseteq G$ and a G-spectrum X, we have maps of G-spectra

$$S^V \underset{\text{fold}}{\overset{\text{pinch}}{\rightleftarrows}} (G/H)_+ \wedge S^V \xrightarrow{\quad\cong\quad} G_+ \underset{H}{\wedge} S^W =: S_W \tag{3}$$

Since for each subgroup $L \subseteq H$,

$$\begin{aligned} \underline{\pi}_W^H X(H/L) &= \underline{\pi}_0^H F(S^W, X)(H/L) \\ &= \underline{\pi}_0^G F(S_W, X)(G/L) \\ &= \underline{\pi}_0^G F((G/H)_+ \wedge S^V, X)(G/L), \end{aligned}$$

the pinch and fold maps induce

$$\underline{\pi}_V^G X \underset{\underline{r}_H^G}{\overset{\underline{t}_H^G}{\rightleftarrows}} \underline{\pi}_W^H X$$

$$\underline{\pi}_V^G X(G/L) \xleftarrow{\underline{t}_H^G(H/L)} \underline{\pi}_W^H X(H/L)$$

$$\underline{\pi}_V^G X(G/K) \xrightarrow[\underline{r}_H^G(G/K)]{} \underline{\pi}_W^H X(H/H \cap K)$$

where the X on the right is the restriction of the X on the left to an H-spectrum, $K \subseteq G$ and $L \subseteq H$. We can conjugate elements on the right by elements of G with the subgroup H acting trivially.

The group action transfer nominally depends on the choice of V that restricts to W, but two such choices V and V' lead to canonically isomorphic groups. For $L \subseteq H \subseteq G$ we have a diagram

$$\begin{array}{ccc} \underline{\pi}_V^G X(G/L) & & \\ \| & \nwarrow^{\underline{t}_H^G(H/L)} & \\ \| & & \underline{\pi}_W^H X(H/L) \\ \| & \swarrow_{\underline{t}_H^G(H/L)} & \\ \underline{\pi}_{V'}^G X(G/L) & & \end{array}$$

The groups on the left are isomorphic because they depend only on the restrictions of V or V' to L. By assumption they have the same restrictions to H.

2. The $RO(G)$-graded homotopy of $H\mathbf{Z}$

We describe part of the $RO(G)$-graded Green functor $\underline{\pi}_\star(H\mathbf{Z})$, where $H\mathbf{Z}$ is the integer Eilenberg-Mac Lane spectrum $H\mathbf{Z}$ in the G-equivariant category, for some cyclic 2-groups G. For each actual (as opposed to virtual) G-representation V we have an equivariant reduced cellular chain complex C_*^V for the space S^V. It is a complex of $\mathbf{Z}[G]$-modules with $H_*(C^V) = H_*(S^{|V|})$.

One can convert such a chain complex C_*^V of $\mathbf{Z}[G]$-modules to one of Mackey functors as follows. Given a $\mathbf{Z}[G]$-module M, we get a Mackey functor $\underline{M}$ defined by

$$\underline{M}(G/H) = M^H \qquad \text{for each subgroup } H \subseteq G. \tag{4}$$

We call this a *fixed point Mackey functor*. When M is a permutation module, meaning the free abelian group on a G-set B, we call $\underline{M}$ a *permutation Mackey functor* [HHR, 2.45]. Given a finite G-CW spectrum X, meaning one built out of cells of the form $G_+ \underset{H}{\wedge} e^n$, we get a reduced cellular chain complex of $\mathbf{Z}[G]$-modules C_*X, leading to a chain complex of fixed point Mackey functors $\underline{C}_*X$. Its homology is a graded Mackey functor $\underline{H}_*X$ with

$$\underline{H}_*X(G/H) = \underline{\pi}_*(X \wedge H\underline{\mathbf{Z}})(G/H) = \pi_*(X \wedge H\underline{\mathbf{Z}})^H.$$

In particular $\underline{H}_*X(G/e) = H_*X$, the underlying homology of X. In general $\underline{H}_*X(G/H)$ is not the same as $H_*(X^H)$ because fixed points do not commute with smash products. We will see an illustration of this below in Example 5, where we will also see that $\underline{H}_*X$ need not be a graded fixed point Mackey functor.

For a finite cyclic 2-group $G = C_{2^k}$, the irreducible representations are the 2-dimensional ones $\lambda(m)$ corresponding to rotation through an angle of $2\pi m/2^k$ for $0 < m < 2^{k-1}$, the sign representation σ and the trivial one of degree one, which we denote by 1. The 2-local homotopy type of $S^{\lambda(m)}$ depends only on the 2-adic valuation of m, so we will only consider $\lambda(2^j)$ for $0 \le j \le k-2$. The planar rotation $\lambda(2^{k-1})$ though angle π is the same representation as 2σ.

We will describe the chain complex C^V for

$$V = a + b\sigma + \sum_{2\le j\le k} c_j\lambda(2^{k-j}).$$

for nonnegative integers a, b and c_j. The isotropy group of V (the largest subgroup fixing all of V) is

$$G_V = \begin{cases} C_{2^k} = G & \text{for } b = c_2 = \cdots = c_k = 0 \\ C_{2^{k-1}} =: G' & \text{for } b > 0 \text{ and } c_2 = \cdots = c_k = 0 \\ C_{2^{k-\ell}} & \text{for } c_\ell > 0 \text{ and } c_{1+\ell} = \cdots = c_k = 0 \end{cases}$$

The sphere S^V has a G-CW structure with reduced cellular chain complex C^V of the form

$$C_n^V = \begin{cases} \mathbf{Z} & \text{for } n = d_0 \\ \mathbf{Z}[G/G'] & \text{for } d_0 < n \le d_1 \\ \mathbf{Z}[G/C_{2^{k-j}}] & \text{for } d_{j-1} < n \le d_j \text{ and } 2 \le j \le \ell \\ 0 & \text{otherwise.} \end{cases}$$

where

$$d_j = \begin{cases} a & \text{for } j = 0 \\ a + b & \text{for } j = 1 \\ a + b + 2c_2 + \cdots + 2c_j & \text{for } 2 \le j \le \ell, \end{cases}$$

so $d_\ell = |V|$.

The boundary map $\partial_n : C_n^V \to C_{n-1}^V$ is determined by the fact that $H_*(C^V) = H_*(S^{|V|})$. More explicitly, let γ be a generator of G and

$$\theta_j = \sum_{0\le t<2^j} \gamma^t \qquad \text{for } 1 \le j \le k.$$

Then we have

$$\partial_n = \begin{cases} \nabla & \text{for } n = 1 + d_0 \\ (1-\gamma)x_n & \text{for } n - d_0 \text{ even and } 2 + d_0 \le n \le d_n \\ x_n & \text{for } n - d_0 \text{ odd } 2 + d_0 \le n \le d_n \\ 0 & \text{otherwise,} \end{cases}$$

where ∇ is the fold map sending $\gamma \mapsto 1$. We will use the same symbol below for the quotient map $\mathbf{Z}[G/H] \to \mathbf{Z}[G/K]$ for $H \subseteq K \subseteq G$. The elements $x_n \in \mathbf{Z}[G]$ for $2 + d_0 \le n \le |V|$ are determined recursively by $x_{2+d_0} = 1$ and

$$x_n x_{n-1} = \theta_j \qquad \text{for } 2 + d_{j-1} < n \le 2 + d_j.$$

It follows that $H_{|V|}C^V = \mathbf{Z}$ generated by either $x_{1+|V|}$ or its product with $1-\gamma$, depending on the parity of b.

This complex is

$$C^V = \Sigma^{|V_0|} C^{V/V_0}$$

where $V_0 = V^G$. This means we can assume without loss of generality that $V_0 = 0$.

An element

$$x \in H_n \underline{C}^V(G/H) = \underline{H}_n S^V(G/H)$$

corresponds to an element $x \in \underline{\pi}_{n-V} H\mathbf{Z}(G/H)$.

We will denote the dual complex $\mathrm{Hom}_{\mathbf{Z}}(C^V, \mathbf{Z})$ by C^{-V}. Its chains lie in dimensions $-n$ for $0 \le n \le |V|$. An element $x \in \underline{H}_{-n}(-V)(G/H)$ corresponds to an element $x \in \underline{\pi}_{V-n} H\mathbf{Z}(G/H)$.

The method we have just described determines only a portion of the $RO(G)$-graded Mackey functor $\underline{\pi}_\star H\underline{\mathbf{Z}}$, namely the groups in which the index differs by an integer from an actual representation V or its negative. For example it does not give us $\underline{\pi}_{\sigma-\lambda(1)} H\underline{\mathbf{Z}}$ for $|G| \ge 4$.

Example 5. The case $G = C_8$ and $V = \sigma + \lambda(1)$. *The representation V is not orientable since it involves an odd multiple of σ. Its unit sphere $S(V)$ is S^2 with the following action of G. There is a generator γ which rotates the equator though an angle of $\pi/4$ while reflecting through the equatorial plane. Thus the poles are fixed by each proper subgroup, and no other point is fixed by a nontrivial subgroup. It follows that in the one point compactification S^V of V we have*

$$(S^V)^H = \begin{cases} S^V & \text{for } H = e \\ S^1 & \text{for } H = C_2 \text{ or } C_4 \\ S^0 & \text{for } H = G \end{cases}$$

$S(V)$ has a G-CW structure with

- *two 0-cells (the north and south poles) interchanged by γ,*
- *eight 1-cells (equally spaced longitudinal lines joining the two poles with alternating orientations) cyclically permuted by γ and*
- *eight 2-cells (regions between two adjacent longitudinal lines) cyclically permuted by γ.*

This means that S^V has a similar G-CW structure with two fxied 0-cells in which each positive dimensioanl cell is the double cone on a cell in $S(V)$. The reduced cellular chain complex C^V is

$$
\begin{array}{ccccccc}
C_0^V & & C_1^V & & C_2^V & & C_3^V \\
\| & & \| & & \| & & \| \\
\mathbf{Z} & \xleftarrow[1]{\nabla} & \mathbf{Z}[G/G'] & \xleftarrow[1]{1-\gamma} & \mathbf{Z}[G] & \xleftarrow[7]{1+\gamma} & \mathbf{Z}[G]
\end{array}
\tag{6}
$$

where $x_2 = 1$, $x_3 = 1+\gamma$ and $x_4 = (1+\gamma^2)(1+\gamma^4)$. The number beneath each arrow indicates its rank as a homomorphism. $H_3 \subseteq C_3$ is the subgroup generated by

$$(1-\gamma)x_4 = (1-\gamma)(1+\gamma^2)(1+\gamma^4).$$

The corresponding chain complex of fixed point Mackey functors is

$$
\begin{array}{ccccccc}
\underline{C}_0^V & & \underline{C}_1^V & & \underline{C}_2^V & & \underline{C}_3^V \\
\mathbf{Z} & \xleftarrow[1]{2} & \mathbf{Z} & \xleftarrow[0]{0} & \mathbf{Z} & \xleftarrow[1]{2} & \mathbf{Z} \\
{\scriptstyle 1}\downarrow\ \uparrow{\scriptstyle 2} & & {\scriptstyle 1+\gamma}\downarrow\ \uparrow{\scriptstyle \nabla} & & {\scriptstyle 1+\gamma}\downarrow\ \uparrow{\scriptstyle \nabla} & & {\scriptstyle 1+\gamma}\downarrow\ \uparrow{\scriptstyle \nabla} \\
\mathbf{Z} & \xleftarrow[1]{\nabla} & \mathbf{Z}[G/G'] & \xleftarrow[1]{4(1-\gamma)} & \mathbf{Z}[G/G'] & \xleftarrow[1]{1+\gamma} & \mathbf{Z}[G/G'] \\
{\scriptstyle 1}\downarrow\ \uparrow{\scriptstyle 2} & & {\scriptstyle 1}\downarrow\ \uparrow{\scriptstyle 2} & & {\scriptstyle 1+\gamma^2}\downarrow\ \uparrow{\scriptstyle \nabla} & & {\scriptstyle 1+\gamma^2}\downarrow\ \uparrow{\scriptstyle \nabla} \\
\mathbf{Z} & \xleftarrow[1]{\nabla} & \mathbf{Z}[G/G'] & \xleftarrow[1]{2(1-\gamma)} & \mathbf{Z}[G/C_2] & \xleftarrow[3]{1+\gamma} & \mathbf{Z}[G/C_2] \\
{\scriptstyle 1}\downarrow\ \uparrow{\scriptstyle 2} & & {\scriptstyle 1}\downarrow\ \uparrow{\scriptstyle 2} & & {\scriptstyle 1+\gamma^4}\downarrow\ \uparrow{\scriptstyle \nabla} & & {\scriptstyle 1+\gamma^4}\downarrow\ \uparrow{\scriptstyle \nabla} \\
\mathbf{Z} & \xleftarrow[1]{\nabla} & \mathbf{Z}[G/G'] & \xleftarrow[1]{1-\gamma} & \mathbf{Z}[G] & \xleftarrow[7]{1+\gamma} & \mathbf{Z}[G]
\end{array}
$$

and its homology is

$$
\begin{array}{cccc}
\underline{H}_0 S^V & \underline{H}_1 S^V & \underline{H}_2 S^V & \underline{H}_3 S^V \\
\mathbf{Z}/2 & 0 & \mathbf{Z}/2 & 0 \\
\downarrow\ \uparrow & \downarrow\ \uparrow & \downarrow\ \uparrow & \downarrow\ \uparrow \\
0 & \mathbf{Z}/4 & 0 & \mathbf{Z}_- \\
\downarrow\ \uparrow & {\scriptstyle 1}\downarrow\ \uparrow{\scriptstyle 2} & \downarrow\ \uparrow & {\scriptstyle 1}\downarrow\ \uparrow{\scriptstyle 2} \\
0 & \mathbf{Z}/2 & 0 & \mathbf{Z}_- \\
\downarrow\ \uparrow & {\scriptstyle 1}\downarrow\ \uparrow & \downarrow\ \uparrow & {\scriptstyle 1}\downarrow\ \uparrow{\scriptstyle 2} \\
0 & 0 & 0 & \mathbf{Z}_-
\end{array}
$$

where $\mathbf{Z}_-$ denotes $\mathbf{Z}[G/H]/(1+\gamma)$ for the appropriate proper subgroup H. In these diagrams of Mackey functors $\underline{M}$, the top and bottom groups are $\underline{M}(G/G)$ and $\underline{M}(G/e)$ with the values of $\underline{M}$ on intermediate groups in between. Downward and upward pointing arrows are restrictions and transfers respectively.

Note that these homology groups are not fixed point Mackey functors, and $\underline{H}_*(G/H)$ is not the same as $H_*(S^V)^H$ for any nontrivial subgroup H.

For the dual spectrum S^{-V} we apply the functor $\mathrm{Hom}_{\mathbf{Z}}(\cdot, \mathbf{Z})$ to (6). The resulting chain complex of fixed point Mackey functors is

$$
\begin{array}{ccccccc}
\underline{C}_0^{-V} & & \underline{C}_{-1}^{-V} & & \underline{C}_{-2}^{-V} & & \underline{C}_{-3}^{-V} \\[1em]
\mathbf{Z} & \xrightarrow{\;1\;} & \mathbf{Z} & \xrightarrow{\;0\;} & \mathbf{Z} & \xrightarrow{\;2\;} & \mathbf{Z} \\
1\downarrow\;\uparrow 2 & & 1+\gamma\downarrow\;\uparrow\nabla & & 1+\gamma\downarrow\;\uparrow\nabla & & 1+\gamma\downarrow\;\uparrow\nabla \\
\mathbf{Z} & \xrightarrow{\;1+\gamma\;} & \mathbf{Z}[G/G'] & \xrightarrow{\;1-\gamma\;} & \mathbf{Z}[G/G'] & \xrightarrow{\;1+\gamma\;} & \mathbf{Z}[G/G'] \\
1\downarrow\;\uparrow 2 & & 1\downarrow\;\uparrow 2 & & 1+\gamma^2\downarrow\;\uparrow\nabla & & 1+\gamma^2\downarrow\;\uparrow\nabla \\
\mathbf{Z} & \xrightarrow{\;1+\gamma\;} & \mathbf{Z}[G/G'] & \xrightarrow{\;(1-\gamma)(1+\gamma^2)\;} & \mathbf{Z}[G/C_2] & \xrightarrow{\;1+\gamma\;} & \mathbf{Z}[G/C_2] \\
1\downarrow\;\uparrow 2 & & 1\downarrow\;\uparrow 2 & & 1+\gamma^4\downarrow\;\uparrow\nabla & & 1+\gamma^4\downarrow\;\uparrow\nabla \\
\mathbf{Z} & \xrightarrow{\;1+\gamma\;} & \mathbf{Z}[G/G'] & \xrightarrow{\;(1-\gamma)x_4\;} & \mathbf{Z}[G] & \xrightarrow{\;1+\gamma\;} & \mathbf{Z}[G]
\end{array}
$$

with homology

$$
\begin{array}{cccc}
\underline{H}_0 S^{-V} & \underline{H}_{-1} S^{-V} & \underline{H}_{-2} S^{-V} & \underline{H}_{-3} S^{-V} \\[1em]
0 & 0 & 0 & \mathbf{Z}/2 \\
\downarrow\;\uparrow & \downarrow\;\uparrow & \downarrow\;\uparrow & 0\downarrow\;\uparrow 1 \\
0 & 0 & 0 & \mathbf{Z}_- \\
\downarrow\;\uparrow & \downarrow\;\uparrow & \downarrow\;\uparrow & 2\downarrow\;\uparrow 1 \\
0 & 0 & 0 & \mathbf{Z}_- \\
\downarrow\;\uparrow & \downarrow\;\uparrow & \downarrow\;\uparrow & 2\downarrow\;\uparrow 1 \\
0 & 0 & 0 & \mathbf{Z}_-
\end{array}
$$

Notice that $\underline{H}_{-3} S^{-V}$ is quite different from $\underline{H}_3 S^V$.

Example 5 illustrates the nonoriented case of the following, whose proof we leave as an exercise.

Proposition 7. The top homology group. *Let G be a finite cyclic 2-group and V a nontrivial representation of G of degree d with $V^G = 0$ and isotropy group G_V. Then $C_d^V = C_{-d}^{-V} = \mathbf{Z}[G/G_V]$ and*

(i) *If V is oriented then $\underline{H}_d S^V = \underline{\mathbf{Z}}$, the constant $\mathbf{Z}$-valued Mackey functor in which each restriction map is an isomorphism and each transfer Tr_H^K is multiplication by $|K/H|$. $\underline{H}_{-d} S^{-V} = \underline{\mathbf{Z}(G, G_V)}$, the constant $\mathbf{Z}$-valued Mackey functor in which*

$$\mathrm{Res}_H^K = \begin{cases} 1 & \text{for } K \subseteq G_V \\ |K/H| & \text{for } G_V \subseteq H \end{cases}$$

and

$$\mathrm{Tr}_H^K = \begin{cases} |K/H| & \text{for } K \subseteq G_V \\ 1 & \text{for } G_V \subseteq H. \end{cases}$$

(These determine all restrictions and transfers.) The functor $\underline{\mathbf{Z}(G,e)}$ is also known as the dual $\underline{\mathbf{Z}}^*$. These isomorphisms are induced by the maps

$$\begin{array}{ccccc} \underline{H}_d S^V & & & & \underline{H}_{-d} S^{-V} \\ \| & & & & \| \\ \underline{\mathbf{Z}} & \xrightarrow{\ \Delta\ } & \underline{\mathbf{Z}[G/G_V]} & \xrightarrow{\ \nabla\ } & \underline{\mathbf{Z}(G,G_V)} \end{array}$$

(ii) *If V is not oriented then $\underline{H}_d S^V = \underline{\mathbf{Z}}_-$, where*

$$\underline{\mathbf{Z}}_-(G/H) = \begin{cases} 0 & \text{for } H = G \\ \mathbf{Z}_- & \text{otherwise} \end{cases}$$

where each restriction map Res_H^K is an isomorphism and each transfer Tr_H^K is multiplication by $|K/H|$ for each proper subgroup K. We also have $\underline{H}_{-d} S^{-V} = \underline{\mathbf{Z}(G,G_V)}_-$, where

$$\underline{\mathbf{Z}(G,G_V)}_-(G/H) = \begin{cases} 0 & \text{for } H = G \text{ and } V = \sigma \\ \mathbf{Z}/2 & \text{for } H = G \text{ and } V \neq \sigma \\ \mathbf{Z}_- & \text{otherwise} \end{cases}$$

with the same restrictions and transfers as $\underline{\mathbf{Z}(G,G_V)}$. These isomorphisms are induced by the evident maps

$$\begin{array}{ccccc} \underline{H}_d S^V & & & & \underline{H}_{-d} S^{-V} \\ \| & & & & \| \\ \underline{\mathbf{Z}}_- & \xrightarrow{\ \Delta_-\ } & \underline{\mathbf{Z}[G/G_V]} & \xrightarrow{\ \nabla_-\ } & \underline{\mathbf{Z}(G,G_V)}_- \end{array}$$

Definition 8. Three elements in $\underline{\pi}_\star^G(H\mathbf{Z})$. *Let V be an actual (as opposed to virtual) representation of the finite cyclic 2-group G with $V^G = 0$ and isotropy group G_V.*

(i) *The equivariant inclusion $S^0 \to S^V$ defines an element in $\underline{\pi}_{-V} S^0(G/G)$ via the isomophisms*

$$\underline{\pi}_{-V} S^0(G/G) = \underline{\pi}_0 S^V(G/G) = \pi_0 S^{V^G} = \pi_0 S^0 = \mathbf{Z},$$

and we will use the symbol a_V to denote its image in $\underline{\pi}_{-V} H\mathbf{Z}(G/G)$.

(ii) *The underlying equivalence $S^V \to S^{|V|}$ defines an element in*

$$\underline{\pi}_V S^{|V|}(G/G_V) = \underline{\pi}_{V-|V|} S^0(G/G_V)$$

and we will use the symbol e_V to denote its image in $\underline{\pi}_{V-|V|} H\mathbf{Z}(G/G_V)$.

(iii) *If W is oriented, there is a map*

$$\Delta : \mathbf{Z} \to C_{|W|}^W$$

as in Proposition 7 giving an element

$$u_W \in \underline{H}_{|W|} S^W(G/G) = \underline{\pi}_{|W|-W} H\mathbf{Z}(G/G).$$

For nonoriented V Proposition 7 gives a map

$$\Delta_- : \mathbf{Z}_- \to C_{|V|}^V$$

and an element

$$u_V \in \underline{H}_{|V|} S^V(G/G') = \underline{\pi}_{|V|-V} H\mathbf{Z}(G/G').$$

Note that a_V and e_V are induced by maps to equivariant spheres while u_W is not. This means that in any spectral sequence based on a filtration where the subquotients are equivariant $H\mathbf{Z}$-modules, elements defined in terms of a_V and e_V will be permanent cycles, while mulitples of u_W can support differentials.

Note also that $a_0 = e_0 = u_0 = 1$. The trivial representations contribute nothing to $\pi_\star(HZ)$. We can limit our attention to representations V with $V^G = 0$. Among such representations of cyclic 2-groups, the oriented ones are precisely the ones of even degree.

Lemma 9. Properties of a_V, e_V and u_W. *The elements $a_V \in \underline{\pi}_{-V} H\mathbf{Z}(G/G)$, $e_V \in \underline{\pi}_{V-|V|} H\mathbf{Z}(G/G_V)$ and $u_W \in \underline{\pi}_{|W|-W} H\mathbf{Z}(G/G)$ for W oriented satisfy the following.*

(i) $a_{V+W} = a_V a_W$ *and* $u_{V+W} = u_V u_W$.
(ii) $|G/G_V| a_V = 0$ *where G_V is the isotropy group of V.*
(iii) *For oriented V, $\mathrm{Tr}_{G_V}^{G}(e_V)$ and $\mathrm{Tr}_{G_V}^{G'}(e_{V+\sigma})$ have infinite order while $\mathrm{Tr}_{G_V}^{G}(e_{V+\sigma})$ has order 2 if $|V| > 0$, and $\mathrm{Tr}_{G_V}^{G}(e_\sigma) = 0$.*
(iv) *For oriented W, $\mathrm{Tr}_{G_V}^{G}(e_W) u_W = |G/G_W| \in \underline{\pi}_0 H\mathbf{Z}(G/G) = \mathbf{Z}$.*
(v) $a_{V+W} \mathrm{Tr}_{G_V}^{G}(e_{V+U}) = 0$ *if* $|V| > 0$.
(vi) *For V and W oriented,* $u_W \mathrm{Tr}_{G_V}^{G}(e_{V+W}) = |G_V/G_{V+W}| \mathrm{Tr}_{G_V}^{G}(e_V)$.
(vii) **The au relation.** *For V and W oriented representations of degree 2 with $G_V \subseteq G_W$, $a_W u_V = |G_W/G_V| a_V u_W$.*

For nonoriented W similar statements hold in $\underline{\pi}_\star H\mathbf{Z}(G/G')$. $2W$ is oriented and u_{2W} is defined in $\underline{\pi}_{2|W|-2W} H\mathbf{Z}(G/G)$ with $\mathrm{Res}_{G'}^{G}(u_{2W}) = u_W^2$.

Proof. (i) This follows from the existence of the pairing $C^V \otimes C^W \to C^{V+W}$. It induces an isomorphism in H_0 and (when both V and W are oriented) in $H_{|V+W|}$.

(ii) This holds because $H_0(V)$ is killed by $|G/G_V|$.

(iii) This follows from Proposition 7.

(iv) Using the Frobenius relation we have

$$\mathrm{Tr}_{G_V}^{G}(e_W) u_W = \mathrm{Tr}_{G_V}^{G}(e_W \mathrm{Res}_1^G(u_W)) = \mathrm{Tr}_{G_V}^{G}(|G/G_V|) = |G/G_V|.$$

(v) We have

$$a_{V+W} \mathrm{Tr}_{G_V}^{G}(e_{V+U}) : S^{-|V|-|U|} \to S^{W-U}.$$

It is null because the bottom cell of S^{W-U} is in dimension $-|U|$.

(vi) Since V is oriented, then we are computing in a torsion free group so we can tensor with the rationals. It follows from (iv) that

$$\begin{aligned} \mathrm{Tr}_{G_{V+W}}^{G}(e_{V+W}) &= \frac{|G/G_{V+W}|}{u_V u_W} \\ \text{and} \qquad \mathrm{Tr}_{G_V}^{G}(e_V) &= \frac{|G/G_V|}{u_V} \\ \text{so} \qquad u_W \mathrm{Tr}_{G_{V+W}}^{G}(e_{V+W}) &= \frac{|G/G_{V+W}|}{u_V} = |G_V/G_{V+W}| \mathrm{Tr}_{G_V}^{G}(e_V). \end{aligned}$$

(vii) The relevant chain complexes are

$$
\begin{array}{cccccccccccc}
 & 0 & & 1 & & 2 & & 3 & & 4 \\
C^V : & \mathbf{Z} & \xleftarrow{\nabla_V} & \mathbf{Z}[G/G_V] & \xleftarrow{1-\gamma} & \mathbf{Z}[G/G_V] & & & & \\
C^W : & \mathbf{Z} & \xleftarrow{\nabla_W} & \mathbf{Z}[G/G_W] & \xleftarrow{1-\gamma} & \mathbf{Z}[G/G_W] & & & & \\
C^{V+W} : & \mathbf{Z} & \xleftarrow{\nabla_W} & \mathbf{Z}[G/G_W] & \xleftarrow{1-\gamma} & \mathbf{Z}[G/G_W] & \xleftarrow{\theta_W} & \mathbf{Z}[G/G_V] & \xleftarrow{1-\gamma} & \mathbf{Z}[G/G_V] \\
\uparrow m & \Vert & & \uparrow m_1 & & \uparrow m_2 & & \uparrow m_3 & & \uparrow m_4 \\
C^V \otimes_{\mathbf{Z}} C^W : & \mathbf{Z} & \xleftarrow{\partial_1} & \begin{array}{c}\mathbf{Z}[G/G_V]\otimes\mathbf{Z} \\ \oplus \\ \mathbf{Z}\otimes\mathbf{Z}[G/G_W]\end{array} & \xleftarrow{\partial_2} & \begin{array}{c}\mathbf{Z}[G/G_V]\otimes\mathbf{Z} \\ \oplus \\ T(V,W) \\ \oplus \\ \mathbf{Z}\otimes\mathbf{Z}[G/G_W]\end{array} & \xleftarrow{\partial_3} & \begin{array}{c}T(V,W) \\ \oplus \\ T(V,W)\end{array} & \xleftarrow{\partial_4} & T(V,W)
\end{array}
$$

where ∇_V and ∇_W are fold or reduction maps sending each power of γ to 1,

$$\theta_W = \sum_{0 \le i < |G/G_W|} \gamma^i$$

and

$$T(V,W) = \mathbf{Z}[G/G_V] \otimes_{\mathbf{Z}} \mathbf{Z}[G/G_W] = \bigoplus_{|G/G_W|} \mathbf{Z}[G/G_V].$$

To describe the maps m_i and ∂_i we use left matrix multiplication on column vectors. We have

$$
\begin{array}{rclcrcl}
\partial_1 & = & \begin{bmatrix} \nabla_V & \nabla_W \end{bmatrix} & \qquad & m_1 & = & \begin{bmatrix} \nabla_{V,W} & 1 \end{bmatrix} \\
\partial_2 & = & \begin{bmatrix} 1-\gamma & ? & 0 \\ 0 & ? & 1-\gamma \end{bmatrix} & \qquad & m_2 & = & \begin{bmatrix} \nabla_{V,W} & ? & 1 \end{bmatrix}
\end{array}
$$

where $\nabla_{V,W}$ is the reduction map and the unidentified maps from $T(V,W)$, namely ∂_3, ∂_4, m_3 and m_4, are not relevant here.

We have a noncommuting diagram

$$
\begin{array}{ccc}
C^V & \xrightarrow{C^V \otimes a_W} & C^V \otimes C^W \\
u_V \uparrow & & \downarrow m_2 \\
\Sigma^2 \mathbf{Z} & & C^{V+W} \\
u_W \downarrow & & \uparrow m_2 \\
C^W & \xrightarrow{a_V \otimes C^W} & C^V \otimes C^W
\end{array}
$$

where the maps to the relevant summands are

$$
\begin{array}{ccc}
\mathbf{Z}[G/G_V] & = & \mathbf{Z}[G/G_V] \otimes \mathbf{Z} \\
\Delta_V \uparrow & & \downarrow \nabla_{V,W} \\
\mathbf{Z} & & \mathbf{Z}[G/G_W] \\
\Delta_W \downarrow & & \Vert \\
\mathbf{Z}[G/G_W] & = & \mathbf{Z} \otimes \mathbf{Z}[G/G_W]
\end{array}
$$

where

$$\Delta_V = \sum_{0\le i<|G/G_V|} \gamma^i$$
$$\Delta_W = \sum_{0\le i<|G/G_W|} \gamma^i.$$

The upper composite is $|G_W/G_V|$ times the lower one, so $a_W u_V = |G_W/G_V| a_V u_W$ as claimed. □

3. The case $G = C_4$

Now let $G = C_4$ with generator γ, and let $G' \subseteq G$ be its index 2 subgroup. Then the above discussion leads at a diagram

$$(10)\qquad \begin{array}{cccccc}
 & RO(G) & & RO(G') & & \mathbf{Z} \\
\mathbf{Z} & \underline{\pi}_\star^G X(G/G) & & & & \\
 & \mathrm{Res}_2^4 \downarrow \uparrow \mathrm{Tr}_2^4 & & & & \\
\mathbf{Z}[G/G'] & \underline{\pi}_\star^G(X)(G/G') & \underset{\underline{r}_2^4}{\overset{\underline{t}_2^4}{\rightleftarrows}} & \underline{\pi}_\star^{G'} X(G'/G') & & \\
 & \mathrm{Res}_1^2 \downarrow \uparrow \mathrm{Tr}_1^2 & & \mathrm{Res}_1^2 \downarrow \uparrow \mathrm{Tr}_1^2 & & \\
\mathbf{Z}[G] & \underline{\pi}_\star^G(X)(G/e) & \underset{\underline{r}_2^4}{\overset{\underline{t}_2^4}{\rightleftarrows}} & \underline{\pi}_\star^{G'} X(G'/e) & \underset{\underline{r}_1^2}{\overset{\underline{t}_1^2}{\rightleftarrows}} & \pi_\star X
\end{array}$$

Here the homotopy groups are modules over the rings shown on the left and graded over the indexing groups shown above. The group action transfers $\underline{t}_2^4$ and $\underline{t}_1^2$ are defined only on groups indexed by representations the smaller group which extend to representations of the larger group. *We will make no use of them in this paper.*

In the bottom row each homotopy group is an underlying homotopy group of X depending only on the degree of the indexing representation. The group action restriction maps are isomorphisms. The group action transfers are

$$\underline{t}_1^2(\underline{r}_1^2(x)) = (1+\gamma^2)x \qquad \text{and} \qquad \underline{t}_2^4(\underline{r}_2^4(y)) = (1+\gamma)y.$$

In the middle row each homotopy group depends only on the restriction of the representation to G'. The restriction $\underline{r}_2^4$ is an isomorphism in each $RO(G)$-graded degree, but it misses half of the $RO(G')$-graded degrees. The transfer is multiplication by $1+\gamma$ when the representation of G' is the restriction of one of G.

We need some notation for Mackey functors to be used in spectral sequence charts. The first four in Table 1 are fixed point Mackey functors (4), meaning they

TABLE 1. Some C_4-Mackey functors

$\square$	$\widehat{\square}$	$\overline{\square}$	$\widehat{\overline{\square}}$	$\bullet$	$\widehat{\bullet}$
$\mathbf{Z}$ $1\downarrow\ \uparrow 2$ $\mathbf{Z}$ $1\downarrow\ \uparrow 2$ $\mathbf{Z}$	$\mathbf{Z}$ $\Delta\downarrow\ \uparrow\nabla$ $\mathbf{Z}[G/G']$ $1\downarrow\ \uparrow 2$ $\mathbf{Z}[G/G']$	0 $\downarrow\ \uparrow$ $\mathbf{Z}_-$ $1\downarrow\ \uparrow 2$ $\mathbf{Z}_-$	0 $\downarrow\ \uparrow$ 0 $\downarrow\ \uparrow$ $\mathbf{Z}[G/G']_-$	$\mathbf{Z}/2$ $\downarrow\ \uparrow$ 0 $\downarrow\ \uparrow$ 0	$\mathbf{Z}/2$ $\Delta\downarrow\ \uparrow\nabla$ $\mathbf{Z}/2[G/G']$ $\downarrow\ \uparrow$ 0
$\dot{\square}$	$\widehat{\dot{\square}}$	$\blacktriangledown$	$\blacktriangle$	◩	$\widehat{◩}$
$\mathbf{Z}/2$ $0\downarrow\ \uparrow 1$ $\mathbf{Z}_-$ $2\downarrow\ \uparrow 1$ $\mathbf{Z}_-$	$\mathbf{Z}/2$ $\Delta\downarrow\ \uparrow\nabla$ $\mathbf{Z}/2[G/G']$ $0\downarrow\ \uparrow 1$ $\mathbf{Z}[G/G']_-$	$\mathbf{Z}/2$ $0\downarrow\ \uparrow 1$ $\mathbf{Z}/2$ $\downarrow\ \uparrow$ 0	$\mathbf{Z}/2$ $1\downarrow\ \uparrow 0$ $\mathbf{Z}/2$ $\downarrow\ \uparrow$ 0	$\mathbf{Z}$ $2\downarrow\ \uparrow 1$ $\mathbf{Z}$ $2\downarrow\ \uparrow 1$ $\mathbf{Z}$	$\mathbf{Z}$ $\Delta\downarrow\ \uparrow\nabla$ $\mathbf{Z}[G/G']$ $2\downarrow\ \uparrow 1$ $\mathbf{Z}[G/G']$
⬓	$\circ$	$\boxed{\blacktriangledown}$	$\widehat{\widehat{\square}}$	$\overline{\bullet}$	$\overline{◩}$
$\mathbf{Z}$ $2\downarrow\ \uparrow 1$ $\mathbf{Z}$ $1\downarrow\ \uparrow 2$ $\mathbf{Z}$	$\mathbf{Z}/4$ $1\downarrow\ \uparrow 2$ $\mathbf{Z}/2$ $\downarrow\ \uparrow$ 0	$\mathbf{Z}/2$ $0\downarrow\ \uparrow 1$ $\mathbf{Z}/2$ $0\downarrow\ \uparrow\nabla$ $\mathbf{Z}[G/G']_-$	$\mathbf{Z}$ $\Delta\downarrow\ \uparrow\nabla$ $\mathbf{Z}[G/G']$ $\Delta\downarrow\ \uparrow\nabla$ $\mathbf{Z}[G]$	0 $\downarrow\ \uparrow$ $\mathbf{Z}/2$ $\downarrow\ \uparrow$ 0	0 $\downarrow\ \uparrow$ $\mathbf{Z}_-$ $2\downarrow\ \uparrow 1$ $\mathbf{Z}_-$

are fixed points of an underlying $Z[G]$-module M, such as

$$\begin{array}{rclcrcl} \mathbf{Z} & = & \mathbf{Z}[G]/(\gamma-1) & \qquad & \mathbf{Z}[G/G'] & = & \mathbf{Z}[G]/(\gamma^2-1) \\ \mathbf{Z}_- & = & \mathbf{Z}[G]/(\gamma+1) & & \mathbf{Z}[G/G']_- & = & \mathbf{Z}[G]/(\gamma^2+1). \end{array}$$

There are short exact sequences

$$0 \longrightarrow \widehat{\bullet} \longrightarrow \widehat{\dot{\square}} \longrightarrow \widehat{\overline{\square}} \longrightarrow 0$$

$$0 \longrightarrow \bullet \longrightarrow \dot{\square} \longrightarrow \overline{\square} \longrightarrow 0$$

$$0 \longrightarrow \blacktriangledown \longrightarrow \circ \longrightarrow \bullet \longrightarrow 0$$

$$0 \longrightarrow \bullet \longrightarrow \circ \longrightarrow \blacktriangle \longrightarrow 0$$

$$0 \longrightarrow ◩ \longrightarrow \square \longrightarrow \circ \longrightarrow 0$$

$$0 \longrightarrow ⬓ \longrightarrow \square \longrightarrow \bullet \longrightarrow 0$$

Here the hat symbol is used for a Mackey functor induced up from C_2, for which our notation is shown in Table 2, where ◩, the dual of $\square$, is the kernel of the surjective map $\square \to \bullet$.

Table 2. Some C_2-Mackey functors

$\square$	$\overline{\square}$	$\bullet$	$◩$	$\dot{\square}$	$\blacktriangledown$	$\blacktriangle$	$\widehat{\square}$
$\mathbf{Z}$	0	$\mathbf{Z}/2$	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}$
$1\downarrow\ \uparrow 2$	$\downarrow\ \uparrow$	$\downarrow\ \uparrow$	$2\downarrow\ \uparrow 1$	$0\downarrow\ \uparrow 1$	$0\downarrow\ \uparrow 1$	$1\downarrow\ \uparrow 0$	$\Delta\downarrow\ \uparrow\nabla$
$\mathbf{Z}$	$\mathbf{Z}_-$	0	$\mathbf{Z}$	$\mathbf{Z}_-$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}[G]$

We have short exact sequences

$$0 \longrightarrow ◩ \longrightarrow \square \longrightarrow \bullet \longrightarrow 0 \tag{11}$$

$$0 \longrightarrow \bullet \longrightarrow \dot{\square} \longrightarrow \overline{\square} \longrightarrow 0 \tag{12}$$

Proposition 13. Exactness of Mackey functor induction. *The induction functor above is exact. It sends a C_2-Mackey functor $\underline{M}$ of the form*

$$\begin{array}{c} \underline{M}(C_2/C_2) \\ \mathrm{Res}_1^2 \big\downarrow \ \big\uparrow \mathrm{Tr}_1^2 \\ \underline{M}(C_2/e) \end{array}$$

to the C_4-Mackey functor $\widehat{\underline{M}}$ of the form

$$\begin{array}{c} \widehat{\underline{M}}(C_4/C_4) = \underline{M}(C_2/C_2) \\ \Delta\otimes\underline{M}(C_2/C_2) \big\downarrow \ \big\uparrow \nabla\otimes\underline{M}(C_2/C_2) \\ \widehat{\underline{M}}(C_4/C_2) = \mathbf{Z}[C_4]\otimes_{\mathbf{Z}[C_2]} \underline{M}(C_2/C_2) \\ \mathbf{Z}[C_4]\otimes_{\mathbf{Z}[C_2]}\mathrm{Res}_1^2 \big\downarrow \ \big\uparrow \mathbf{Z}[C_4]\otimes_{\mathbf{Z}[C_2]}\mathrm{Tr}_1^2 \\ \widehat{\underline{M}}(C_4/e) = \mathbf{Z}[C_4]\otimes_{\mathbf{Z}[C_2]} \underline{M}(C_2/e) \end{array}$$

The same holds for induction up to G of a Mackey functor defined for a subgroup H of G for any finite G.

Definition 14. A $\mathbf{Z}[C_4]$-enriched C_2-Mackey functor. *For a C_2-Mackey functor $\underline{M}$ as above, $\widetilde{\underline{M}}$ will denote the C_2-Mackey functor enriched over $\mathbf{Z}[C_4]$ defined by*

$$\widetilde{\underline{M}}(C_2/H) = \mathbf{Z}[C_4]\otimes_{\mathbf{Z}[C_2]} \underline{M}(C_2/H)$$

for $H = C_2$ or e with structure maps as above.

4. Some chain complexes of Mackey functors

As noted above, a G-CW complex X, meaning one built out of cells of the form $G_+ \underset{H}{\wedge} e^n$, has a reduced cellular chain complex of $\mathbf{Z}[G]$-modules C_*X, leading to a chain complex of fixed point Mackey functors (see (4)) $\underline{C}_*X$. When $X = S^V$ for a representation V, we will denote this complexes $\underline{C}_*^V$. Its homology is the graded Mackey functor $\underline{H}_*X$. Here we will apply the methods of §2 to three examples.

(i) Let $G = C_2$ with generator γ, and $X = S^{n\rho}$ for $n > 0$, where ρ denotes the regular representation. We have seen before [HHR, 3.6] that it has a reduced cellular chain complex C with

$$C_i^{n\rho_2} = \begin{cases} \mathbf{Z}[G]/(\gamma - 1) & \text{for } i = n \\ \mathbf{Z}[G] & \text{for } n < i \leq 2n \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Let c_i denote a generator of C_i. The boundary operator d is given by

$$d(c_{i+1}) = \begin{cases} c_i & \text{for } i = n \\ \gamma_{i+1-n}(c_i) & \text{for } n < i \leq 2n \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

where $\gamma_i = 1 - (-1)^i\gamma$. For future reference, let

$$\epsilon_i = 1 - (-1)^i = \begin{cases} 0 & \text{for } i \text{ even} \\ 2 & \text{for } i \text{ odd.} \end{cases}$$

This chain complex has the form

$$\begin{array}{ccccccccc}
n & & n+1 & & n+2 & & n+3 & & 2n \\
\square & \xleftarrow{\nabla} & \widehat{\square} & \xleftarrow{\gamma_2} & \widehat{\square} & \xleftarrow{\gamma_3} & \widehat{\square} & \longleftarrow \cdots \xleftarrow{\gamma_n} & \widehat{\square} \\
\mathbf{Z} & \xleftarrow{2} & \mathbf{Z} & \xleftarrow{0} & \mathbf{Z} & \xleftarrow{2} & \mathbf{Z} & \longleftarrow \cdots \xleftarrow{\epsilon_n} & \mathbf{Z} \\
{\scriptstyle 1}\downarrow\uparrow{\scriptstyle 2} & & {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla} & & {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla} & & {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla} & & {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla} \\
\mathbf{Z} & \xleftarrow{\nabla} & \mathbf{Z}[G] & \xleftarrow{\gamma_2} & \mathbf{Z}[G] & \xleftarrow{\gamma_3} & \mathbf{Z}[G] & \longleftarrow \cdots \xleftarrow{\gamma_n} & \mathbf{Z}[G]
\end{array}$$

Passing to homology we get

$$\begin{array}{cccccc}
n & n+1 & n+2 & n+3 & & 2n \\
\bullet & 0 & \bullet & 0 & \cdots & \underline{H}_{2n} \\
\mathbf{Z}/2 & 0 & \mathbf{Z}/2 & 0 & \cdots & \underline{H}_{2n}(G/G) \\
\downarrow\uparrow & \downarrow\uparrow & \downarrow\uparrow & \downarrow\uparrow & & {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla} \\
0 & 0 & 0 & 0 & \cdots & \mathbf{Z}[G]/(\gamma_{n+1})
\end{array}$$

where

$$\underline{H}_{2n}(G/G) = \begin{cases} \mathbf{Z} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases} \quad \text{and} \quad \underline{H}_{2n} = \begin{cases} \square & \text{for } n \text{ even} \\ \overline{\square} & \text{for } n \text{ odd} \end{cases}$$

Here $\square$ and $\overline{\square}$ are fixed point Mackey functors but $\bullet$ is not.

Similar calculations can be made for $S^{n\rho_2}$ for $n < 0$. The results are indicated in Figure 1.

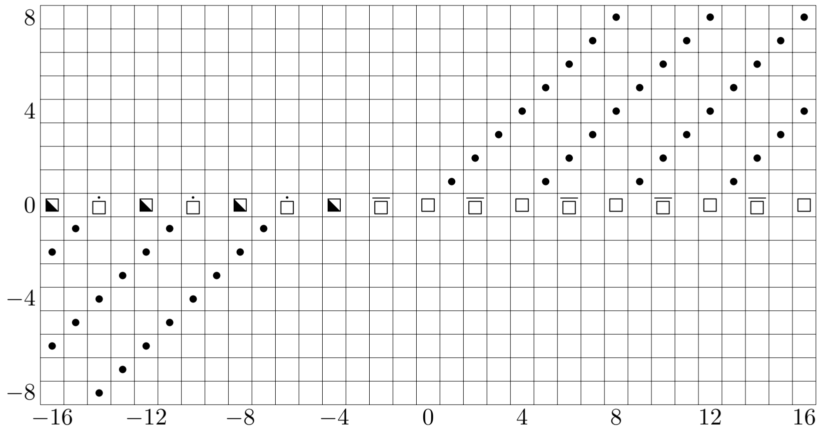


FIGURE 1. The (collapsing) Mackey functor slice spectral sequence for $H\underline{\mathbf{Z}} \wedge \bigvee_{n\in\mathbf{Z}} S^{n\rho_2}$. The symbols are defined in Table 2.

In other words the $RO(G)$-graded Mackey functor valued homotopy of $H\underline{\mathbf{Z}}$ is as follows. For $n > -3/2$ we have

$$\underline{\pi}_i \Sigma^{n\rho_2} H\underline{\mathbf{Z}} = \underline{\pi}_{i-n\rho_2} H\underline{\mathbf{Z}} = \begin{cases} \square & \text{for } n \text{ even and } i = 2n \\ \bullet & \text{for } n \text{ even and } i = 2n-2j \text{ with } 0 < j \leq n/2 \\ \overline{\square} & \text{for } n \text{ odd and } i = 2n \\ \bullet & \text{for } n \text{ odd and } i = 2n+1-2j \\ & \qquad \text{with } 0 < j \leq (n+1)/2 \\ 0 & \text{otherwise} \end{cases}$$

For $n < -3/2$ we have

$$\underline{\pi}_i \Sigma^{n\rho_2} H\underline{\mathbf{Z}} = \underline{\pi}_{i-n\rho_2} H\underline{\mathbf{Z}} = \begin{cases} \text{◩} & \text{for } n \text{ even and } i = 2n \\ \bullet & \text{for } n \text{ even and } i = 2n+2j-1 \\ & \qquad \text{with } 0 < j \leq (-3-n)/2 \\ \dot{\square} & \text{for } n \text{ odd and } i = 2n \\ \bullet & \text{for } n \text{ odd and } i = 2n+2j \\ & \qquad \text{with } 0 < j \leq (-3-n)/2 \\ 0 & \text{otherwise} \end{cases}$$

We can use Definition 8 to name some elements of these groups.

Note that $H\underline{\mathbf{Z}}$ is a commutative ring spectrum, so there is a commutative multiplication in $\underline{\pi}_\star H\underline{\mathbf{Z}}$, making it a commutative Green functor. For such a functor $\underline{M}$ on a general group G, the restriction maps are a ring homomorphisms while the transfer maps satisfy the Frobenius relations (1).

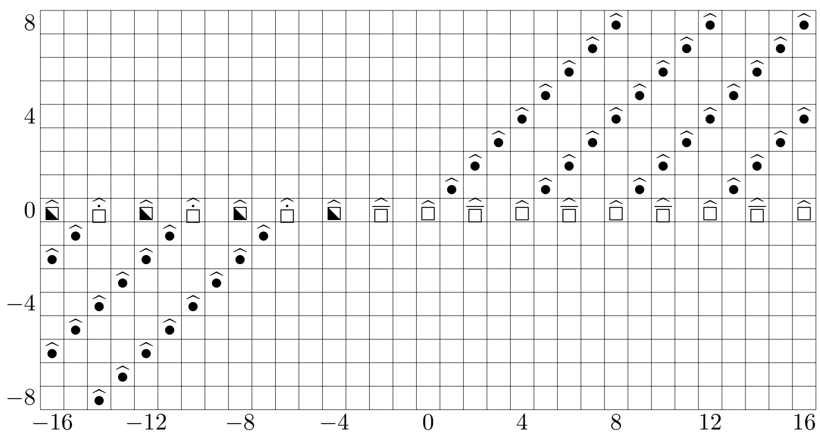


FIGURE 2. The Mackey functor slice spectral sequence for $G_+ \wedge_{G'} \bigvee_{n \in \mathbf{Z}} \Sigma^{n\rho_2} H\underline{\mathbf{Z}}$, for $G = C_4$ and $G' = C_2$. The symbols are for C_4-Mackey functors defined in Table 1.

Then

$$
(17)\qquad \left\{
\begin{array}{rcl}
\text{2-SLICE:} & & \\
a = a_\sigma & \in & \underline{\pi}_1 \Sigma^{\rho_2} H\underline{\mathbf{Z}}(G/G) = \underline{\pi}_{-\sigma} H\underline{\mathbf{Z}}(G/G) \\
x = u_\sigma & \in & \underline{\pi}_2 \Sigma^{\rho_2} H\underline{\mathbf{Z}}(G/e) = \underline{\pi}_{1-\sigma} H\underline{\mathbf{Z}}(G/e) \\
 & & \qquad \text{with } \gamma(x) = -x \\
\text{4-SLICE:} & & \\
u = u_{2\sigma} & \in & \underline{\pi}_4 \Sigma^{2\rho_2} H\underline{\mathbf{Z}}(G/G) = \underline{\pi}_{2-2\sigma} H\underline{\mathbf{Z}}(G/G) \\
 & & \qquad \text{with } \mathrm{Res}(u) = x^2 \\
\text{NEGATIVE SLICES:} & & \\
z_n = e_{2n\rho_2} & \in & \underline{\pi}_{-4n} \Sigma^{-2n\rho_2} H\underline{\mathbf{Z}}(G/e) \\
 & = & \underline{\pi}_{2n(\sigma-1)} H\underline{\mathbf{Z}}(G/e) \qquad \text{for } n > 0 \\
a^{-i}\mathrm{Tr}(x^{-2n-1}) & \in & \underline{\pi}_{-4n-2-i} \Sigma^{-(2n+1+i)\rho_2} H\underline{\mathbf{Z}}(G/G) \\
 & = & \underline{\pi}_{(2n+1)(\sigma-1)+i\sigma} H\underline{\mathbf{Z}}(G/G) \\
 & & \qquad \text{for } n > 0 \text{ and } i \geq 0
\end{array}
\right.
$$

are the generators of their respective groups. We have relations

$$
\begin{array}{rclcrcl}
2a & = & 0 & \qquad & \mathrm{Res}(a) & = & 0 \\
z_n & = & x^{-2n} & & \mathrm{Tr}(x^n) & = & \begin{cases} 2u^{n/2} & \text{for } n \text{ even and } n \geq 0 \\ \mathrm{Tr}(z_{-n/2}) & \text{for } n \text{ even and } n < 0 \\ 0 & \text{for } n \text{ odd and } n > -3. \end{cases}
\end{array}
$$

(ii) Let $G = C_4$ with generator γ, $G' = C_2 \subseteq G$, the subgroup generated by γ^2, and $\widehat{S}(n, G') = G_+ \underset{G'}{\wedge} S^{n\rho_2}$. Thus we have

$$
C_*(\widehat{S}(n, G')) = \mathbf{Z}[G] \otimes_{\mathbf{Z}[G']} C_*^{n\rho_2}
$$

with $C_*^{n\rho_2}$ as in (15). The calculations of the previous example carry over verbatim by the exactness of 13.

The results are indicated in Figure 2, which is obtained from the Figure 1 by putting a hat over each symbol. We will name elements in the groups shown here as follows. For an element

$$\alpha \in \underline{\pi}_{i-V}^{G'} H\underline{\mathbf{Z}}(G'/K) = \underline{\pi}_i^{G'} \Sigma^V H\underline{\mathbf{Z}}(G'/K),$$

such as those listed in (17), we denote the two corresponding elements in

$$\begin{aligned} \underline{\pi}_i^G(G_+ \underset{G'}{\wedge} \Sigma^V H\underline{\mathbf{Z}})(G/K) &= \mathbf{Z}[G] \otimes_{\mathbf{Z}[G']} \underline{\pi}_i^{G'} \Sigma^V H\underline{\mathbf{Z}}(G'/K) \\ &= \underline{\pi}_{i-V}^{G'} H\underline{\mathbf{Z}}(G'/K) \oplus \underline{\pi}_{i-V}^{G'} H\underline{\mathbf{Z}}(G'/K) \end{aligned}$$

by $\widehat{\alpha}$ and $\gamma(\widehat{\alpha})$. We have $\gamma^2(\widehat{\alpha}) = \pm\widehat{\alpha}$, and there is no canonical choice of $\widehat{\alpha}$.

When the representation V of G' is the restriction of a representation W of G, then this group is

$$\underline{\pi}_{i-W}^G(G_+ \underset{G'}{\wedge} H\underline{\mathbf{Z}})(G/K)$$

When $K = G'$, the two elements have the same image in

$$\underline{\pi}_i^G(G_+ \underset{G'}{\wedge} \Sigma^V H\underline{\mathbf{Z}})(G/G) = \pi_i(G_+ \underset{G'}{\wedge} \Sigma^V H\underline{\mathbf{Z}})^G = \pi_i(\Sigma^V H\underline{\mathbf{Z}})^{G'} = \underline{\pi}_i(\Sigma^V H\underline{\mathbf{Z}})(G'/G')$$

under the transfer, namely the element corresponding to α under the evident isomorphisms. Hence if α (or a set of such elements) generates the group $\underline{\pi}_i^{G'} \Sigma^V H\underline{\mathbf{Z}}(G'/G')$, then $t_2^4(\widehat{\alpha})$ (or the corresponding set) generates $\underline{\pi}_i^G(G_+ \underset{G'}{\wedge} \Sigma^V H\underline{\mathbf{Z}})(G/G)$.

(iii) Let $G = C_4$ and $X = S^{n\rho_4}$. Then the reduced cellular chain complex is

$$C_i^{n\rho_4} = \begin{cases} \mathbf{Z} & \text{for } i = n \\ \mathbf{Z}[G/G'] & \text{for } n < i \le 2n \\ \mathbf{Z}[G] & \text{for } 2n < i \le 4n \\ 0 & \text{otherwise} \end{cases}$$

with

$$d(c_{i+1}) = \begin{cases} c_i & \text{for } i = n \\ \gamma_{i+1-n} c_i & \text{for } n < i \le 2n \\ \theta_{i+1-n} c_i & \text{for } 2n < i < 4n \text{ and } i \text{ even} \\ \gamma_{i+1-n} c_i & \text{for } 2n < i < 4n \text{ and } i \text{ odd} \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\theta_i = \gamma_i(1+\gamma^2) = (1-(-1)^i\gamma)(1+\gamma^2).$$

The fixed point Mackey functors for $\mathbf{Z} = \mathbf{Z}[G/G]$, $\mathbf{Z}[G/G']$ and $\mathbf{Z}[G] = \mathbf{Z}[G/e]$, are $\square$, $\widehat{\square}$ and $\widehat{\widehat{\square}}$. In low dimensions the chain complex of Mackey functors is

$$\begin{array}{ccccccccc} n & & n+1 & & n+2 & & n+3 & & \\ \square & \xleftarrow{\nabla} & \widehat{\square} & \xleftarrow{1-\gamma} & \widehat{\square} & \xleftarrow{1+\gamma} & \widehat{\square} & \longleftarrow & \cdots \end{array}$$

In homology this gives

$$\begin{array}{ccccc} n & n+1 & n+2 & n+3 & \\ \bullet & 0 & \bullet & 0 & \cdots \end{array}$$

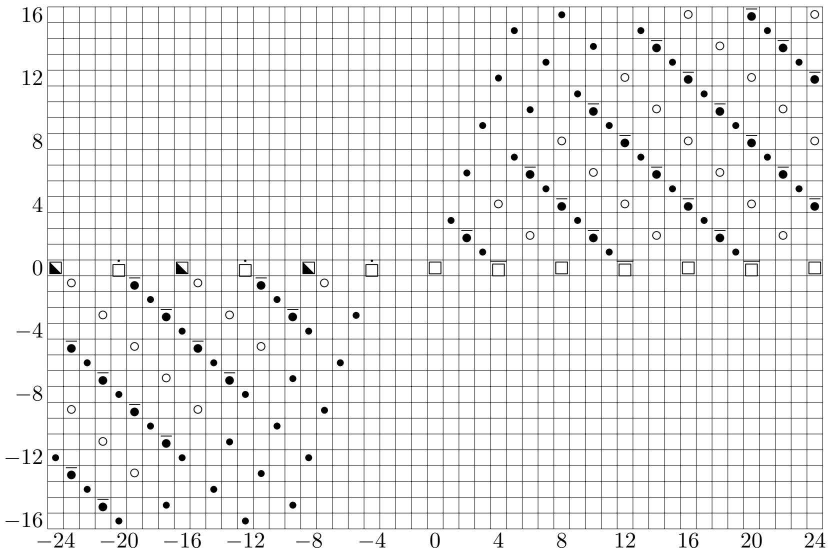
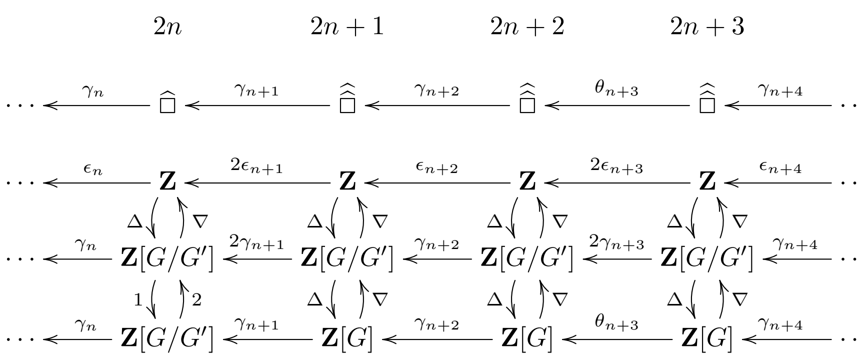


FIGURE 3. The Mackey functor slice spectral sequence for $\bigvee_{n\in\mathbf{Z}} \Sigma^{n\rho_4} H\underline{\mathbf{Z}}$. The symbols are defined in Table 1. The Mackey functor at position $(4n-s, s)$ is $\underline{\pi}_{n(4-\rho_4)-s} H\mathbf{Z}$.

In dimensions near $2n$ we have



The homology is

$$\underline{H}_{2n+i} = \begin{cases} \circ & \text{for } n \text{ and } i \text{ even and } 0 \le i < 2n \\ \bullet & \text{for } n \text{ and } i \text{ odd and } 0 \le i < 2n \\ \bar{\bullet} & \text{for } n \text{ odd and } i \text{ even and } 0 \le i < 2n \\ \Box & \text{for } n \text{ even and } i = 2n \\ \overline{\Box} & \text{for } n \text{ odd and } i = 2n \\ 0 & \text{otherwise} \end{cases}$$

Again similar calculations can be made for $S^{n\rho_4}$ for $n < 0$. The results are indicated in Figure 3. The Mackey functors in filtration 0 (the horizontal axis) are the ones described in Proposition 7.

As in (i), we name some of these elements. Let $G = C_4$ and $G' = C_2 \subseteq G$. Recall that the regular representation ρ_4 is $1+\sigma+\lambda$ where σ is the sign representation and λ is the 2-dimensional representation given by a rotation of order 4.

Note that while Figure 1 shows all of $\underline{\pi}_\star H\underline{\mathbf{Z}}$ for $G = C_2$, Figure 3 shows only a bigraded portion of this trigraded Mackey functor for $G = C_4$, namely the groups for which the index differs by an integer from a multiple of ρ_4. We will need to refer to some elements not shown in the latter chart, namely

$$\begin{array}{rclrcl} a_\sigma & \in & \underline{\pi}_{-\sigma} H\underline{\mathbf{Z}}(G/G) & a_\lambda & \in & \underline{\pi}_{-\lambda} H\underline{\mathbf{Z}}(G/G) \\ u_\sigma & \in & \underline{\pi}_{1-\sigma} H\underline{\mathbf{Z}}(G/G') & u_\lambda & \in & \underline{\pi}_{2-\lambda} H\underline{\mathbf{Z}}(G/G) \\ u_{2\sigma} & \in & \underline{\pi}_{2-2\sigma} H\underline{\mathbf{Z}}(G/G) & & & \end{array}$$

with $2a_\lambda u_{2\sigma} = a_\sigma^2 u_\lambda$ and $\mathrm{Res}_2^4(u_{2\sigma}) = u_\sigma^2$; see Definition 8 and Lemma 9.

We will denote the generator of $\underline{E}_2^{s,t}(G/H)$ by $x_{t-s,s}$, $y_{t-s,s}$ and $z_{t-s,s}$ for $H = G$, G' and e respectively. Then the generators for the groups in the 4-slice are

$$\begin{array}{rcl} y_{4,0} = u_{\rho_4} = u_\sigma \mathrm{Res}_2^4(u_\lambda) & \in & \underline{\pi}_4 \Sigma^{\rho_4} H\underline{\mathbf{Z}}(G/G') = \underline{\pi}_{3-\sigma-\lambda} H\underline{\mathbf{Z}}(G/G') \\ & & \text{with } \gamma(x_{4,0}) = -x_{4,0} \\ x_{3,1} = a_\sigma u_\lambda & \in & \underline{\pi}_3 \Sigma^{\rho_4} H\underline{\mathbf{Z}}(G/G) = \underline{\pi}_{2-\sigma-\lambda} H\underline{\mathbf{Z}}(G/G) \\ y_{2,2} = \mathrm{Res}_2^4(a_\lambda) u_\sigma & \in & \underline{\pi}_2 \Sigma^{\rho_4} H\underline{\mathbf{Z}}(G/G') = \underline{\pi}_{1-\sigma-\lambda} H\underline{\mathbf{Z}}(G/G') \\ x_{1,3} = a_{\rho_4} = a_\sigma a_\lambda & \in & \underline{\pi}_1 \Sigma^{\rho_4} H\underline{\mathbf{Z}}(G/G) = \underline{\pi}_{-\sigma-\lambda} H\underline{\mathbf{Z}}(G/G) \end{array}$$

and the ones for the 8-slice are

$$\begin{array}{rcl} x_{8,0} = u_{2\lambda+2\sigma} = u_{2\rho_4} & \in & \underline{\pi}_8 \Sigma^{2\rho_4} H\underline{\mathbf{Z}}(G/G) = \underline{\pi}_{6-2\sigma-2\lambda} H\underline{\mathbf{Z}}(G/G) \\ & & \text{with } y_{4,0}^2 = y_{8,0} = \mathrm{Res}_2^4(x_{8,0}) \\ x_{6,2} = a_\lambda u_{\lambda+2\sigma} & \in & \underline{\pi}_6 \Sigma^{2\rho_4} H\underline{\mathbf{Z}}(G/G) = \underline{\pi}_{4-2\sigma-2\lambda} H\underline{\mathbf{Z}}(G/G) \\ & & \text{with } x_{3,1}^2 = 2x_{6,2} \\ & & \text{and } y_{4,0} y_{2,2} = y_{6,2} = \mathrm{Res}_2^4(x_{6,2}) \\ x_{4,4} = a_{2\lambda} u_{2\sigma} & \in & \underline{\pi}_4 \Sigma^{2\rho_4} H\underline{\mathbf{Z}}(G/G) = \underline{\pi}_{2-2\sigma-2\lambda} H\underline{\mathbf{Z}}(G/G) \\ & & \text{with } y_{2,2}^2 = y_{4,4} = \mathrm{Res}_2^4(x_{4,4}) \\ & & \text{and } x_{1,3} x_{3,1} = 2x_{4,4} \\ x_{6,2} = x_{3,1}^2 & \in & \underline{\pi}_2 \Sigma^{2\rho_4} H\underline{\mathbf{Z}}(G/G) = \underline{\pi}_{-2\sigma-2\lambda} H\underline{\mathbf{Z}}(G/G). \end{array}$$

These elements and their restrictions generate $\underline{\pi}_* \Sigma^{m\rho_4} H\underline{\mathbf{Z}}$ for $m = 1$ and 2. For $m > 2$ the groups are generated by products of these elements. There are relations

$$\begin{array}{rclrclrcl} 2a_\sigma & = & 0 & \mathrm{Res}_2^4 a_\sigma & = & 0 & & & \\ 4a_\lambda & = & 0 & 2\mathrm{Res}_2^4 a_\lambda & = & 0 & \mathrm{Res}_1^4 a_\lambda & = & 0 \end{array}$$

The element

$$z_{4,0} = \mathrm{Res}_1^2(y_{4,0}) = \mathrm{Res}_1^2(u_{\rho_4}) \in \underline{\pi}_4 \Sigma^{\rho_4} H\underline{\mathbf{Z}}(G/e)$$

is invertible with $\gamma(y) = -y$, $z_{4,0}^2 = z_{8,0} = \mathrm{Res}_1^4(x_{8,0})$ and

$$z_{-4m,0} := z_{4,0}^{-m} = e_{m\rho_4} \in \underline{\pi}_{-4m} \Sigma^{-m\rho_4} H\underline{\mathbf{Z}}(G/e) \qquad \text{for } m > 0.$$

These elements and their transfers generate the groups in

$$\underline{\pi}_{-4m} \Sigma^{-m\rho_4} H\underline{\mathbf{Z}} \qquad \text{for } m > 0.$$

Proof???

Theorem 18. Divisibilities in the negative regular slices for C_4. *There are the following infinite divisibilities in the third quadrant of the spectral sequence in Figure 3.*

- $x_{-4,0} = \mathrm{Tr}_1^4(z_{-4,0})$ *is infinitely divisible by* $x_{4,4}$ *and* $x_{1,3}$*, meaning that*
$$x_{4,4}^j x_{1,3}^k x_{-4-4j-k,-4j-3k} = x_{-4,0} \qquad \text{for } j,k \geq 0.$$
- $x_{-7,-1}$ *is infinitely divisible by* $x_{4,4}$*,* $x_{6,2}$ *and* $x_{8,0}$*, meaning that*
$$x_{4,4}^i x_{6,2}^j x_{8,0}^k x_{-7-4i-6j-8k,-1-4i-2j} = x_{-7,-1} \qquad \text{for } i,j,k \geq 0,$$
subject to the relation $x_{6,2}^2 = x_{8,0}x_{4,4}$.
- $x_{-10,-2}$ *is infinitely divisible by* $x_{4,4}$*,* $x_{6,2}$ *and* $x_{8,0}$*, meaning that*
$$x_{4,4}^i x_{6,2}^j x_{8,0}^k x_{-10-4i-6j-8k,-2-4i-2j} = x_{-10,-2} \qquad \text{for } i,j,k \geq 0$$
with
$$\begin{aligned} 2x_{-7-4i-6j-8k,-1-4i-2j} &= x_{3,1}x_{-10-4i-6j-8k,-2-4i-2j} \\ &= x_{1,3}x_{-8-4i-8j-6k,-4-4i-2j} \\ &\qquad \text{for } i,j,k \geq 0. \end{aligned}$$
- $y_{-7,-1} = \mathrm{Res}_2^4(x_{-7,-1})$ *is infinitely divisible by* $y_{2,2}$ *and* $y_{4,0}$*, meaning that*
$$y_{2,2}^j y_{4,0}^k y_{-7-2j-4k,-1-2j} = y_{-7,-1} \qquad \text{for } j,k \geq 0.$$

5. The C_4-spectrum $k_{\mathbf{H}}$

Before defining our spectrum we need to recall some formulas from [HHR]. Let G be a finite cyclic 2-group with generator γ. In [HHR, 5.47] we defined generators

$$\overline{r}_k = \overline{r}_k^G \in \underline{\pi}_{k\rho_2}^{C_2} MU^{((G))}(C_2/C_2) \tag{19}$$

(note that this group is a module over G/C_2) and

$$r_k = \underline{r}_1^2 \mathrm{Res}_1^2(\overline{r}_k) \in \pi_{n\rho_2}^e MU^{((G))}(e/e) = \pi_{2n}^u MU^{((G))}.$$

These are defined in terms of the coefficients

$$\overline{m}_k \in \underline{\pi}_{k\rho_2}^{C_2} H\underline{\mathbf{Z}}_{(2)} \wedge MU^{((G))}(C_2/C_2)$$

of the logarithm of the formal group associated with the left unit map from MU to $MU^{((G))}$. For small k we have

$$\begin{aligned} \overline{r}_1 &= (1-\gamma)(\overline{m}_1) \\ \overline{r}_2 &= \overline{m}_2 - 2\gamma(\overline{m}_1)(1-\gamma)(\overline{m}_1) \\ \overline{r}_3 &= (1-\gamma)(\overline{m}_3) - \gamma(\overline{m}_1)(\overline{m}_1^2 + 2\overline{m}_1\gamma(\overline{m}_1) - 3\gamma(\overline{m}_1)^2 - 2\overline{m}_2) \end{aligned}$$

Now let $G = C_4$ and $G' = C_2 \subseteq G$. The generators $\overline{r}_k^G$ are the $\overline{r}_k$ defined above. We also have generators $\overline{r}_k^{G'}$ defined by similar formulas with γ replaced by γ^2; recall that $\gamma^2(\overline{m}_k) = (-1)^k\overline{m}_k$. Thus we have

$$\begin{aligned} \overline{r}_1^{G'} &= 2\overline{m}_1 \\ \overline{r}_2^{G'} &= \overline{m}_2 + 4\overline{m}_1^2 \\ \overline{r}_3^{G'} &= 2\overline{m}_3 - 2\overline{m}_1\overline{m}_2 - 4\overline{m}_1^3 \end{aligned}$$

If we set $\overline{r}_2 = 0$ and $\overline{r}_3 = 0$, we get

$$(20)\qquad \begin{cases} \overline{r}_1^{G'} &= (1+\gamma)(\overline{r}_1) \\ \overline{r}_2^{G'} &= 3\overline{r}_1\gamma(\overline{r}_1) + \gamma(\overline{r}_1)^2 \\ \overline{r}_3^{G'} &= 5\overline{r}_1^2\gamma(\overline{r}_1) + 5\overline{r}_1\gamma(\overline{r}_1)^2 + \gamma(\overline{r}_1)^3 \\ \overline{r}_3^{G'}\gamma(\overline{r}_3^{G'}) &= -\overline{r}_1\gamma(\overline{r}_1)\left(5\gamma(\overline{r}_1)^2 - 5\overline{r}_1\gamma(\overline{r}_1) + \overline{r}_1^2\right) \\ & \qquad \left(\gamma(\overline{r}_1)^2 + 5\overline{r}_1\gamma(\overline{r}_1) + 5\overline{r}_1^2\right) \\ &= -5\overline{r}_1^5\gamma(\overline{r}_1) + 20\overline{r}_1^4\gamma(\overline{r}_1)^2 - \overline{r}_1^3\gamma(\overline{r}_1)^3 \\ & \qquad -20\overline{r}_1^2\gamma(\overline{r}_1)^4 - 5\overline{r}_1\gamma(\overline{r}_1)^5 \end{cases}$$

Let $k_{\mathbf{H}}$ be the G-spectrum obtained from $MU^{((G))}$ by killing the r_ns and their conjugates for $n \geq 2$. We will often use a (second) subscript ϵ to indicate the action of γ, so $\gamma(x_\epsilon) = x_{1+\epsilon}$ and $x_{2+\epsilon} = \pm x_\epsilon$.

Then we have

(21)
$$\pi_*^u k_{\mathbf{H}} = \underline{\pi}_* k_{\mathbf{H}}(G/e) = \mathbf{Z}[r_1,\, \gamma(r_1)] = \mathbf{Z}[r_{1,0},\, r_{1,1}] \qquad \text{where } \gamma^2(r_{1,\epsilon}) = -r_{1,\epsilon}.$$

Here we use $r_{1,\epsilon}$ and $\overline{r}_{1,\epsilon}$ to denote the images of elements of the same name in the homotopy of $MU^{((G))}$.

The Periodicity Theorem [HHR, 9.12] states that inverting a class

$$D \in \underline{\pi}_{4\rho_4} k_{\mathbf{H}}(G/G)$$

whose image under $\underline{r}_2^4\mathrm{Res}_2^4$ is divisible by $\overline{r}_{3,0}^{G'}\overline{r}_{3,1}^{G'}$ (see (20)) and $\overline{r}_{1,0}\overline{r}_{1,1}$ makes $u_{8\rho_4}$ a permanent cycle. Let

$$D = (\underline{r}_2^4\mathrm{Res}_2^4)^{-1}\left(\overline{r}_{1,0}^{G}\overline{r}_{1,1}^{G}\overline{r}_3^{G'}\gamma(\overline{r}_3^{G'})\right) \in \underline{\pi}_{4\rho_4} k_{\mathbf{H}}(G/G)$$

(see Table 3 for a more explicit description) and $K_{\mathbf{H}} = D^{-1}k_{\mathbf{H}}$. Then we know that $\Sigma^{32}K_{\mathbf{H}}$ is equivalent to $K_{\mathbf{H}}$.

The Slice and Reduction Theorems [HHR, 6.1 and 6.5] imply that the $2k$th slice of $k_{\mathbf{H}}$ is the $2k$th wedge summand of

$$H\underline{\mathbf{Z}} \wedge N_2^4\left(\bigvee_{i\geq 0} S^{i\rho_2}\right).$$

It follows that over G' the $2k$th slice is a wedge of $k+1$ copies of $H\underline{\mathbf{Z}} \wedge S^{k\rho_2}$.

The group $\underline{\pi}_{\rho_2}^{G'} k_{\mathbf{H}}(G'/e)$ is *not* in the image of the group action restriction $\underline{r}_2^4$ because ρ_2 is not the restriction of a representation of G. However, $\pi_2^u k_{\mathbf{H}}$ is refined (in the sense of [HHR, 5.29]) by a map from

$$(22)\qquad S_{\rho_2} = G_+ \underset{G'}{\wedge} S^{\rho_2} \xrightarrow{\overline{s}_1} k_{\mathbf{H}}.$$

The reduction theorem implies that the 2-slice $P_2^2 k_{\mathbf{H}}$ is $S_{\rho_2} \wedge H\underline{\mathbf{Z}}$. We know that

$$\underline{\pi}_2(S_{\rho_2} \wedge H\underline{\mathbf{Z}}) = \widehat{\square}.$$

We use the symbols r_1 and $\gamma(r_1)$ to denote the generators of the underlying abelian group of $\widehat{\square}(G/e) = \mathbf{Z}[G/G']_-$. These elements have trivial fixed point transfers and

$$\underline{\pi}_2(S_{\rho_2} \wedge H\underline{\mathbf{Z}})(G/G') = 0.$$

Tables 3 and 4 describe some elements in the low dimensional homotopy of $k_{\mathbf{H}}$, which we now discuss.

Given an element in $\pi_\star MU^{((G))}$, we will often use the same symbol to denote its image in $\pi_\star k_{\mathbf{H}}$. For example, in [HHR, 9.1]

$$\bar{\mathfrak{d}}_k \in \pi^G_{(2^k-1)\rho_4} MU^{((G))} = \underline{\pi}^G_{(2^k-1)\rho_4} MU^{((G))}(G/G) \tag{23}$$

was defined to be the composite

$$S^{(2^k-1)\rho_4} = N_2^4 S^{(2^k-1)\rho_2} \xrightarrow{N_2^4 \bar{r}_{2^k-1}} N_2^4 MU^{((G))} \longrightarrow MU^{((G))}.$$

We will use the same symbol to denote its image in $\underline{\pi}^G_{\rho_4} k_{\mathbf{H}}(G/G)$.

The element $\eta \in \pi_1 S^0$ (coming from the Hopf map $S^3 \to S^2$) has image $a_\sigma \bar{r}_1 \in \underline{\pi}_1^{G'} k_{\mathbf{R}}(G'/G')$. There are two corresponding elements

$$\eta_\epsilon \in \underline{\pi}_1^{G'} k_{\mathbf{H}}(G'/G') \qquad \text{for } \epsilon = 0, 1.$$

We use the same symbol for their preimages under $\underline{r}_2^4$. We denote by η again the image of either under the transfer Tr_2^4. It is the image of the Hopf map in $\underline{\pi}_1 k_{\mathbf{H}}(G/G)$, and $\mathrm{Res}_2^4(\eta) = \eta_0 + \eta_1$.

Its cube is killed by a d_3 in the slice spectral sequence, as is the sum of any two monomials of degree 3 in the η_ϵ. It follows that in $\underline{E}_4$ each such monomial is equal to η_0^3. It has a nontrivial transfer, which we denote by x_3.

In [HHR, 5.51] we defined

$$f_k = a_{\bar{\rho}}^k N_2^g \bar{r}_k \in \underline{\pi}_k MU^{((G))}(G/G) \tag{24}$$

Proof??? for a finite cyclic 2-group G. Its slice filtration is $k(g-1)$ and we conjecture that

$$\mathrm{Tr}_{G'}^G(u_\sigma \mathrm{Res}^G_{G'}(x)) = a_\sigma f_1 x. \tag{25}$$

Somehow this is related to the first slice differential,

$$d_{1+|G|}(u_{2\sigma}) = a_\sigma^2 f_1.$$

In particular, for $G = C_4$ we have

$$f_1 = a_\sigma a_\lambda \bar{\mathfrak{d}}_1 \qquad \text{with } \mathrm{Tr}_2^4(u_\sigma \mathrm{Res}_2^4(x)) = a_\sigma f_1 x.$$

For example

$$\mathrm{Tr}_2^4(\eta_0\eta_1) = \mathrm{Tr}_2^4(u_\sigma \mathrm{Res}_2^4(a_\lambda \bar{\mathfrak{d}}_1)) = a_\sigma f_1 a_\lambda \bar{\mathfrak{d}}_1 = f_1^2.$$

The Hopf element $\nu \in \pi_3 S^0$ has image

$$a_\sigma u_\lambda \bar{\mathfrak{d}}_1 \in \underline{\pi}_3 k_{\mathbf{H}}(G/G),$$

Proof??? so we also denote the latter by ν. It has an exotic restriction η_0^3 (filtration jump two), which implies that

$$2\nu = \mathrm{Tr}_2^4(\mathrm{Res}_2^4(\nu)) = \mathrm{Tr}_2^4(\eta_0^3) = x_3.$$

One way to see this is to use the Periodicity Theorem to equate $\underline{\pi}_3 k_{\mathbf{H}}$ with $\underline{\pi}_{-29} k_{\mathbf{H}}$, which can be shown to be the Mackey functor $\circ$ in slice filtration -32. Another argument not relying on periodicity is given below in (39).

The exotic restriction on ν implies

$$\mathrm{Res}_2^4(\nu^2) = \eta_0^6,$$

with filtration jump 4.

Theorem 26. The Hurewicz image *The elements* $\eta \in \underline{\pi}_1 k_{\mathbf{H}}(G/G)$, $\nu \in \underline{\pi}_3 k_{\mathbf{H}}(G/G)$, $\epsilon \in \underline{\pi}_8 k_{\mathbf{H}}(G/G)$, $\kappa \in \underline{\pi}_{14} k_{\mathbf{H}}(G/G)$, *and* $\overline{\kappa} \in \underline{\pi}_{20} k_{\mathbf{H}}(G/G)$ *are the images of elements of the same names in* $\pi_* S^0$.

We refer the reader to [Rav86, Table A3.3] for more information about these elements.

Proof. Suppose we know this for ν and $\overline{\kappa}$. Then $\Delta_1^{-4}\nu$ is represented by an element of filtration -3 whose product with ν^2 is nontrivial. This implies that ν^3 has nontrivial image in $\underline{\pi}_9 k_{\mathbf{H}}(G/G)$. This is a nontrivial multiplicative extension in the first quadrant, but not in the third.

Since $\nu^3 = \eta\epsilon$ in $\pi_* S^0$, this implies that η and ϵ are both detected and have the images stated in Table 4. It follows that $\epsilon\overline{\kappa}$ has nontrivial image here. Since $\kappa^2 = \epsilon\overline{\kappa}$ in $\pi_* S^0$, κ must also be detected. Its only possible image is the one indicated. More details needed here.

Both ν and $\overline{\kappa}$ have images of order 8 in $\pi_* tmf$ and its $K(2)$ localization. The latter is the homotopy fixed point set of an action of the binary tetrahedral group G_{24} acting on E_2. This in turn is a retract of the homotopy fixed point set of the quaternion group Q_8. A restriction and transfer argument shows that both elements have order at least 4 in the homotopy fixed point set of $C_4 \subset Q_8$. WE NEED TO RELATE THIS C_4 ACTION TO THE ONE WE ARE STUDYING. □

6. Slices for $k_{\mathbf{H}}$ and $K_{\mathbf{H}}$

Let

$$\begin{array}{rcll} \Delta_1 & = & u_{2\rho_4}\bar{\mathfrak{d}}_1^2 & \in \underline{\pi}_8 k_{\mathbf{H}}(G/G) \\ \delta_1 & = & u_{\rho_4}\mathrm{Res}_2^4(\bar{\mathfrak{d}}_1) & \in \underline{\pi}_4 k_{\mathbf{H}}(G/G'), \end{array}$$

so $\delta_1^2 = \mathrm{Res}_2^4(\Delta_1)$. Hence we have

$$\begin{array}{rcl} \delta_1^m & = & \delta_1^{m-2[m/2]}\mathrm{Res}_2^4(\Delta_1^{[m/2]}) \\ & = & \left\{ \begin{array}{ll} u_\sigma \mathrm{Res}_2^4(u_{2\sigma}^{[m/2]} u_\lambda^m \bar{\mathfrak{d}}_1^m) & \text{for } m \text{ odd} \\ \mathrm{Res}_2^4(u_{2\sigma}^{m/2} u_\lambda^m \bar{\mathfrak{d}}_1^m) & \text{for } m \text{ even.} \end{array} \right. \end{array}$$

Theorem 27. The slice $\underline{E}_2$-term for $k_{\mathbf{H}}$. *The slices of* $k_{\mathbf{H}}$ *are*

$$P_s^s k_{\mathbf{H}} = \left\{ \begin{array}{ll} \bigvee_{0 \le m \le s/4} X_{m,s/2-m} & \text{for } s \text{ even and } s \ge 0 \\ * & \text{otherwise} \end{array} \right.$$

where

$$X_{m,n} = \left\{ \begin{array}{ll} \Sigma^{m\rho_4} H\underline{\mathbf{Z}} & \text{for } m = n \\ G_+ \underset{G'}{\wedge} \Sigma^{(m+n)\rho_2} H\underline{\mathbf{Z}} & \text{for } m < n \end{array} \right.$$

TABLE 3. Some elements of filtration 0 in the homotopy of and slice spectral sequence for $k_{\mathbf{H}}$. Refer to Table 4 for targets of differentials and transfers.

Element	Description
$\overline{r}_{1,\epsilon} \in \underline{\pi}^{G'}_{\rho_2} k_{\mathbf{H}}(G'/G')$ with $\overline{r}_{1,2} = -\overline{r}_{1,0}$	Images from (19) defined in [HHR, 5.47]
$r_{1,\epsilon} \in \underline{\pi}^{G'}_{2} k_{\mathbf{H}}(G'/e)$	$u_{\sigma_2}\mathrm{Res}^2_1(\overline{r}_{1,\epsilon})$
$r_{1,\epsilon} \in \underline{\pi}^{G}_{2} k_{\mathbf{H}}(G/e)$ with $r_{1,2} = -r_{1,0}$	Preimages of the above under $\underline{r}^4_2$, generating $\widehat{\overline{\square}} = \underline{\pi}^G_2 k_{\mathbf{H}}/\mathrm{torsion}$
$\overline{s}_{2,\epsilon} \in \underline{\pi}^{G}_{\rho_4} k_{\mathbf{H}}(G/G')$	Preimages of $\overline{r}^2_{1,\epsilon}$ under $\underline{r}^4_2$
$\bar{\mathfrak{d}}_1 \in \underline{\pi}^{G}_{\rho_4} k_{\mathbf{H}}(G/G)$ with $\underline{r}^4_2\mathrm{Res}^4_2(\bar{\mathfrak{d}}_1) = \overline{r}_{1,0}\overline{r}_{1,1}$ and $\mathrm{Tr}^4_2(\mathrm{Res}^4_2(\bar{\mathfrak{d}}_1)) = 0$	Image from (23) defined in [HHR, 9.1]
$\bar{t}_2 \in \underline{\pi}^{G}_{\rho_4} k_{\mathbf{H}}(G/G)$ with $\mathrm{Res}^4_2(\bar{t}_2) = \overline{s}_{2,0} - \overline{s}_{2,1}$	$(-1)^\epsilon \mathrm{Tr}^4_2(\overline{s}_{2,\epsilon})$
$D \in \underline{\pi}_{4\rho_4} k_{\mathbf{H}}(G/G)$, the periodicity element	$\bar{\mathfrak{d}}_1^2(-5\bar{t}_2^2 + 20\bar{t}_2\bar{\mathfrak{d}}_1 + 9\bar{\mathfrak{d}}_1^2)$
$u_\sigma \in \underline{\pi}_{1-\sigma} k_{\mathbf{H}}(G/G')$ with $\gamma(u_\sigma) = -u_\sigma$ and $\mathrm{Tr}^4_2(u_\sigma) = a_\sigma f_1$ (exotic transfer).	Isomorphic image of $1 \in \underline{\pi}_0 k_{\mathbf{H}}(G/G')$
$\Sigma_{2,\epsilon} \in \underline{E}_2^{0,4} k_{\mathbf{H}}(G/G')$ with $\Sigma_{2,2} = \Sigma_{2,0}$ and $d_3(\Sigma_{2,\epsilon}) = \eta_\epsilon^2(\eta_0 + \eta_1)$	$(-1)^\epsilon u_{\rho_4}\overline{s}_{2,\epsilon}$ $= (-1)^\epsilon u_\sigma \mathrm{Res}^4_2(u_\lambda)\overline{s}_{2,\epsilon}$
$T_2 \in \underline{E}_2^{0,4} k_{\mathbf{H}}(G/G)$ with $\mathrm{Res}^4_2(T_2) = \Sigma_{2,0} + \Sigma_{2,1}$ and $d_3(T_2) = \eta^3$	$\mathrm{Tr}^4_2(\Sigma_{2,\epsilon}) = (-1)^\epsilon u_\lambda \mathrm{Tr}^4_2(u_\sigma \overline{s}_{2,\epsilon})$
$T_4 \in \underline{E}_2^{0,8} k_{\mathbf{H}}(G/G)$ with $T_4^2 = \Delta_1(T_2^2 - 4\Delta_1)$, $\mathrm{Res}^4_2(T_4) = (\Sigma_{2,0} - \Sigma_{2,1})\delta_1$ and $d_3(T_4) = 0$	$(-1)^\epsilon \mathrm{Tr}^4_2(\Sigma_{2,\epsilon}\delta_1) = u_{2\sigma}u_\lambda^2 \bar{t}_2\bar{\mathfrak{d}}_1$
$\delta_1 \in \underline{E}_2^{0,4} k_{\mathbf{H}}(G/G')$ with $\gamma(\delta_1) = -\delta_1$, $\mathrm{Tr}^4_2(\delta_1) = 0$ and $d_3(\delta_1) = \eta_0\eta_1(\eta_0 + \eta_1)$	$u_{\rho_4}\mathrm{Res}^4_2(\bar{\mathfrak{d}}_1) = u_\sigma \mathrm{Res}^4_2(u_\lambda \bar{\mathfrak{d}}_1)$
$\Delta_1 \in \underline{E}_2^{0,8} k_{\mathbf{H}}(G/G)$ with $\mathrm{Res}^4_2(\Delta_1) = \delta_1^2$ and $d_5(\Delta_1) = \nu x_4$	$u_{2\rho_4}\bar{\mathfrak{d}}_1^2 = u_{2\sigma}u_\lambda^2\bar{\mathfrak{d}}_1^2$

The structure of $\pi_*^u k_{\mathbf{H}}$ as a $\mathbf{Z}[G]$-module (see (21)) leads to four types of orbits and slices:

(1) $\left\{(r_{1,0}r_{1,1})^{2\ell}\right\}$ *leading to $X_{2\ell,2\ell}$ for $\ell \geq 0$; see the leftmost diagonal in Figure 5. On the 0-line we have a copy of $\square$ (see Table 1) generated under restrictions by*

$$\Delta_1^\ell = u_{2\ell\rho_4}\bar{\mathfrak{d}}_1^{2\ell} = u_{2\sigma}^\ell u_\lambda^{2\ell}\bar{\mathfrak{d}}_1^{2\ell} \in \underline{E}_2^{0,8\ell}(G/G).$$

In positive filtrations we have

$$\circ \quad \subseteq \quad \underline{E}_2^{2j,8\ell} \qquad \text{generated by}$$

TABLE 4. Some elements of positive filtration in the homotopy of and slice spectral sequence for $k_{\mathbf{H}}$.

Element	Description
$\eta_\epsilon \in \underline{\pi}_1^{G'} k_{\mathbf{H}}(G'/G')$ with $2\eta_\epsilon = 0$	$a_{\sigma_2}\overline{r}_{1,\epsilon}$
$\eta_\epsilon \in \underline{\pi}_1^{G} k_{\mathbf{H}}(G/G')$	Preimage of the above under $\underline{r}_2^4$, generating the summand $\widehat{\bullet}$ of $\underline{\pi}_1 k_{\mathbf{H}}$
$f_1 \in \underline{\pi}_1 k_{\mathbf{H}}(G/G)$	$a_\sigma a_\lambda \bar{\mathfrak{d}}_1$, generating the summand $\bullet$ of $\underline{\pi}_1 k_{\mathbf{H}}$
$\eta \in \underline{\pi}_1^{G} k_{\mathbf{H}}(G/G)$ with $\mathrm{Res}_2^4(\eta) = \eta_0 + \eta_1 \in \underline{\pi}_1^{G} k_{\mathbf{H}}(G/G')$	$\mathrm{Tr}_2^4(\eta_\epsilon) + f_1$
η_ϵ^2, $\eta_0\eta_1 \in \underline{\pi}_2^{G} k_{\mathbf{H}}(G/G')$ with $\mathrm{Tr}_2^4(\eta_\epsilon^2) = \eta^2$ and $\mathrm{Tr}_2^4(\eta_0\eta_1) = f_1^2$ (exotic transfer)	$u_\sigma \mathrm{Res}_2^4(a_\lambda)\overline{s}_{2,\epsilon}$ and $u_\sigma \mathrm{Res}_2^4(a_\lambda \bar{\mathfrak{d}}_1)$, generating the torsion $\widehat{\bullet} \oplus \blacktriangledown$ in $\underline{\pi}_2^{G} k_{\mathbf{H}}$
$\eta_0^3 = \eta_0^2\eta_1 = \eta_0\eta_1^2 = \eta_1^3 \in \underline{\pi}_3^{G} k_{\mathbf{H}}(G/G')$	$\eta_\epsilon u_\sigma \mathrm{Res}_2^4(a_\lambda \bar{\mathfrak{d}}_1) = \eta_\epsilon u_\sigma \mathrm{Res}_2^4(a_\lambda \overline{s}_{2,\epsilon})$
$x_3 \in \underline{\pi}_3 k_{\mathbf{H}}(G/G)$ with $\mathrm{Res}_2^4(x_3) = 0$	$\mathrm{Tr}_2^4(\eta_0^3)$
$\nu \in \underline{\pi}_3 k_{\mathbf{H}}(G/G)$ with $\mathrm{Res}_2^4(\nu) = \eta_0^3$ and $2\nu = x_3$ (exotic restriction and group extension)	$a_\sigma u_\lambda \bar{\mathfrak{d}}_1$, generating $\circ = \underline{\pi}_3 k_{\mathbf{H}}$
$x_4 \in \underline{E}_2^{4,8}(G/G)$ with $d_5(x_4) = f_1^3$, $\mathrm{Res}_2^4(x_4) = (\eta_0\eta_1)^2 = \eta_0^4$ and $2x_4 = f_1\nu$	$a_\lambda^2 u_{2\sigma} \bar{\mathfrak{d}}_1^2$
$\nu^2 \in \underline{\pi}_6 k_{\mathbf{H}}(G/G)$	$2a_\lambda u_\lambda u_{2\sigma} \bar{\mathfrak{d}}_1^2 = \langle 2, \eta, f_1, f_1^2 \rangle$
$\epsilon \in \underline{\pi}_8 k_{\mathbf{H}}(G/G)$	Represents $x_4^2 \in \underline{E}_2^{8,16}(G/G)$
$\nu^3 = \eta\epsilon \in \underline{\pi}_9 k_{\mathbf{H}}(G/G)$	Represents $f_1 x_4^2 \in \underline{E}_2^{8,16}(G/G)$
$\kappa \in \underline{\pi}_{14} k_{\mathbf{H}}(G/G)$	$2a_\lambda u_{2\sigma}^2 u_\lambda^3 \bar{\mathfrak{d}}_1^4$
$\overline{\kappa} \in \underline{\pi}_{20} k_{\mathbf{H}}(G/G)$	$a_\lambda^2 u_{2\sigma}^3 u_\lambda^4 \bar{\mathfrak{d}}_1^6$

$$
\begin{array}{rcll}
a_\lambda^j u_{2\sigma}^\ell u_\lambda^{2\ell-j} \bar{\mathfrak{d}}_1^{2\ell} & \in & \underline{E}_2^{2j,8\ell}(G/G) & \textit{for } 0 < j \le 2\ell \textit{ and} \\
\bullet & \subseteq & \underline{E}_2^{2k+4\ell,8\ell} & \textit{generated by} \\
a_\sigma^{2k} a_\lambda^{2\ell} u_{2\sigma}^{\ell-k} \bar{\mathfrak{d}}_1^{2\ell} & \in & \underline{E}_2^{2k+4\ell,8\ell}(G/G) & \textit{for } 0 < k \le \ell.
\end{array}
$$

(2) $\{(r_{1,0}r_{1,1})^{2\ell+1}\}$ *leading to* $X_{2\ell+1,2\ell+1}$ *for* $\ell \ge 0$; *see the leftmost diagonal in Figure 6. On the 0-line we have a copy of* $\overline{\square}$ *generated under restrictions by*

$$\delta_1^{2\ell+1} = u_\sigma^{2\ell+1} \mathrm{Res}_2^4(u_\lambda \bar{\mathfrak{d}}_1)^{2\ell+1} \quad \in \quad \underline{E}_2^{0,8\ell+4}(G/G').$$

In positive filtrations we have

$$
\begin{array}{rcll}
\overline{\bullet} & \subseteq & \underline{E}_2^{2j,8\ell+4} & \textit{generated by} \\
u_\sigma^{2\ell+1} \mathrm{Res}_2^4(a_\lambda^j u_\lambda^{2\ell+1-j} \bar{\mathfrak{d}}_1^{2\ell+1}) & \in & \underline{E}_2^{2j,8\ell+4}(G/G') & \textit{for } 0 < j \le 2\ell+1, \\
\bullet & \subseteq & \underline{E}_2^{2j+1,8\ell+4} & \textit{generated by} \\
a_\sigma a_\lambda^j u_{2\sigma}^\ell u_\lambda^{2\ell+1-j} \bar{\mathfrak{d}}_1^{2\ell+1} & \in & \underline{E}_2^{2j+1,8\ell+4}(G/G) & \textit{for } 0 \le j \le 2\ell+1 \textit{ and} \\
\bullet & \subseteq & \underline{E}_2^{2k+4\ell+3,8\ell+4} & \textit{generated by} \\
a_\sigma^{2k+1} a_\lambda^{2\ell+1} u_{2\sigma}^{\ell-k} \bar{\mathfrak{d}}_1^{2\ell+1} & \in & \underline{E}_2^{2k+4\ell+3,8\ell+4}(G/G) & \textit{for } 0 < k \le \ell.
\end{array}
$$

(3) $\{r_{1,0}^i r_{1,1}^{2\ell-i}, r_{1,0}^{2\ell-i} r_{1,1}^i\}$ *leading to* $X_{i,2\ell-i}$ *for* $0 \le i < \ell$*; see other diagonals in Figure 5. On the 0-line we have a copy of* $\widehat{\square}$ *generated (under* Tr_2^4, Res_1^2 *and the group action) by*

$$u_\sigma^\ell \bar{s}_2^{\ell-i} \mathrm{Res}_2^4(u_\lambda^\ell \bar{\mathfrak{d}}_1^i) \in \underline{E}_2^{0,4\ell}(G/G')$$

In positive filtrations we have

$$\begin{aligned} \widehat{\bullet} &\subseteq \underline{E}_2^{2j,4\ell} \qquad \text{generated by} \\ u_\sigma^\ell \bar{s}_2^{\ell-i} \mathrm{Res}_2^4(a_\lambda^j u_\lambda^{\ell-j} \bar{\mathfrak{d}}_1^i) & \\ &\in \underline{E}_2^{2j,4\ell}(G/G') \qquad \text{for } 0 < j \le \ell \\ &= \eta_\epsilon^{2j} u_\sigma^{\ell-j} \bar{s}_2^{\ell-i-j} \mathrm{Res}_2^4(u_\lambda^{\ell-j} \bar{\mathfrak{d}}_1^i) \qquad \text{for } 0 < j < \ell - i. \end{aligned}$$

(4) $\{r_{1,0}^i r_{1,1}^{2\ell+1-i}, r_{1,0}^{2\ell+1-i} r_{1,1}^i\}$ *leading to* $X_{i,2\ell+1-i}$ *for* $0 \le i \le \ell$*; see other diagonals in Figure 6. On the 0-line we have a copy of* $\widehat{\widehat{\square}}$ *generated (under transfers and the group action) by*

$$r_{1,0} \mathrm{Res}_1^2(u_\sigma^\ell \bar{s}_2^{\ell-i}) \mathrm{Res}_1^4(u_\lambda^\ell \bar{\mathfrak{d}}_1^i) \in \underline{E}_2^{0,4\ell+2}(G/e)$$

In positive filtrations we have

$$\begin{aligned} \widehat{\bullet} &\subseteq \underline{E}_2^{2j+1,4\ell+2} \qquad \text{generated by} \\ \eta_\epsilon u_\sigma^\ell \bar{s}_2^{\ell-i} \mathrm{Res}_2^4(a_\lambda^j u_\lambda^{\ell-j} \bar{\mathfrak{d}}_1^i) & \\ &\in \underline{E}_2^{2j+1,4\ell+2}(G/G') \qquad \text{for } 0 \le j \le \ell \\ &= \eta_\epsilon^{2j+1} u_\sigma^{\ell-j} \bar{s}_2^{\ell-i-j} \mathrm{Res}_2^4(u_\lambda^{\ell-j} \bar{\mathfrak{d}}_1^i) \qquad \text{for } 0 \le j \le \ell - i. \end{aligned}$$

Corollary 28. A subring of the slice E_2-term. *The ring* $\underline{E}_2 k_{\mathbf{H}}(G/G')$ *is (see Tables 3 and 4)*

$$\mathbf{Z}[\delta_1, \Sigma_{2,\epsilon}, \eta_\epsilon \colon \epsilon = 0, 1] / \left(2\eta_\epsilon, \delta_1^2 - \Sigma_{2,0}\Sigma_{2,1}, \eta_\epsilon \Sigma_{2,\epsilon+1} + \eta_{1+\epsilon}\delta_1\right).$$

In particular the elements η_0 *and* η_1 *are algebraically independent mod 2 with*

$$\gamma^\epsilon(\eta_0^m \eta_1^n) \in \underline{\pi}_{m+n} X_{m,n}(G/G') \qquad \text{for } m \le n.$$

The element $(\eta_0\eta_1)^2$ *is the fixed point restriction of*

$$u_{2\sigma} a_\lambda^2 \bar{\mathfrak{d}}_1^2 \in \underline{E}_2^{4,8} k_{\mathbf{H}}(G/G),$$

which has order 4, and the transfer of the former is twice the latter. The element $\eta_0\eta_1$ *is not in the image of* Res_2^4 *and has trivial transfer in* $\underline{E}_2$.

Proof. We detect this subring with the monomorphism

$$\begin{aligned} \underline{E}_2 k_{\mathbf{H}}(G/G') &\xrightarrow{r_2^4} \underline{E}_2 k_{\mathbf{H}}(G'/G') \\ \eta_\epsilon &\longmapsto a_\sigma \bar{r}_{1,\epsilon} \\ \Sigma_{2,\epsilon} &\longmapsto u_{2\sigma} \bar{r}_{1,\epsilon}^2 \\ \delta_1 &\longmapsto u_{2\sigma} \bar{r}_{1,0} \bar{r}_{1,1}, \end{aligned}$$

in which all the relations are transparent. ∎

Corollary 29. Slices for $K_{\mathbf{H}}$. *The slices of $k_{\mathbf{H}}$ are*

$$P_s^s k_{\mathbf{H}} = \begin{cases} \bigvee_{m\le s/4} X_{m,s/2-m} & \text{for } s \text{ even and } s\ge 0 \\ * & \text{otherwise} \end{cases}$$

where $X_{m,n}$ is as in Theorem 27. Here m can be any integer, and we still require that $m \le n$.

Proof. Recall that $K_{\mathbf{H}}$ is obtained from $k_{\mathbf{H}}$ by inverting a certain element $D \in \pi_{4\rho_4} k_{\mathbf{H}}$ described in Table 3. Thus $K_{\mathbf{H}}$ is the homotopy colimit of the diagram

$$k_{\mathbf{H}} \xrightarrow{D} \Sigma^{-4\rho_4} k_{\mathbf{H}} \xrightarrow{D} \Sigma^{-8\rho_4} k_{\mathbf{H}} \xrightarrow{D} \cdots$$

Desuspending by $4\rho_4$ converts slices to slices, so for even s we have

$$\begin{aligned} P_s^s K_{\mathbf{H}} &= \lim_{k\to\infty} \Sigma^{-4k\rho_4} P_{s+16k}^{s+16k} k_{\mathbf{H}} \\ &= \lim_{k\to\infty} \Sigma^{-4k\rho_4} \bigvee_{0\le m\le s/4+8k} X_{m,s/2+8k-m} \\ &= \lim_{k\to\infty} \bigvee_{0\le m\le s/4+4k} X_{m-4k,s/2+4k-m} \\ &= \lim_{k\to\infty} \bigvee_{-4k\le m\le s/4} X_{m,s/2-m} \\ &= \bigvee_{m\le s/4} X_{m,s/2-m}. \quad \square \end{aligned}$$

7. Generalities on differentials

Now we turn to differentials. Our starting point is the Slice Differentials Theorem of [HHR, 9.9], which says that in the slice spectral sequence for $MU^{((G))}$ for an arbitrary finite cyclic 2-group G of order g, the first nontrivial differential on various powers of $u_{2\sigma}$ is

$$(30) \qquad d_r(u_{2\sigma}^{2^{k-1}}) = a_\sigma^{2^k} a_{\overline{\rho}}^{2^k-1} N_2^g(\overline{r}_{2^k-1}^G) \in \underline{E}_r^{r,r+2^k(1-\sigma)-1} MU^{((G))}(G/G),$$

where $r = 1 + (2^k - 1)g$ and $\overline{\rho}$ is the reduced regular representation of G. In particular

$$(31) \qquad \begin{cases} d_3(u_{2\sigma}) = a_\sigma^3 \overline{r}_1 \in \underline{E}_3^{3,4-2\sigma} MU_{\mathbf{R}}(G/G) & \text{for } G = C_2 \\ d_5(u_{2\sigma}) = a_\sigma^3 a_\lambda \bar{\mathfrak{d}}_1 \in \underline{E}_5^{5,6-2\sigma} MU^{((G))}(G/G) & \text{for } G = C_4 \\ d_7(u_{2\sigma}^2) = a_\sigma^7 \overline{r}_3 \in \underline{E}_3^{7,10-4\sigma} MU_{\mathbf{R}}(G/G) & \text{for } G = C_2. \end{cases}$$

Now, as before, let $G = C_4$ and $G' = C_2 \subseteq G$. We need to translate the d_3 above in the slice spectral sequence for $MU_{\mathbf{R}}$ into a statement about the one for $k_{\mathbf{H}}$. We have an equivariant multiplication map m of G'-spectra

$$\begin{array}{ccccc}
 & & MU^{((G))} & & \\
 & & \| & & \\
MU_{\mathbf{R}} & \xrightarrow{\eta_L} & MU_{\mathbf{R}} \wedge MU_{\mathbf{R}} & \xrightarrow{m} & MU_{\mathbf{R}} \\
\overline{r}_1^{G'} & \longmapsto & \overline{r}_{1,0}^G + \overline{r}_{1,1}^G & \longmapsto & \overline{r}_1^{G'} \\
 & & a_\sigma^3(\overline{r}_{1,0}^G + \overline{r}_{1,1}^G) & \longmapsto & a_\sigma^3 \overline{r}_1^{G'} \\
\overline{r}_3^{G'} & \longmapsto & \left(\begin{array}{c} 5\overline{r}_{1,0}^G \overline{r}_{1,1}^G(\overline{r}_{1,0}^G + \overline{r}_{1,1}^G) \\ +(\overline{r}_{1,1}^G)^3 \bmod (\overline{r}_2^G, \overline{r}_3^G) \end{array} \right) & \longmapsto & \overline{r}_3^{G'}
\end{array}$$

where the elements lie in $\underline{\pi}_{\rho_2}^{G'}(\cdot)(G'/G')$. In the slice spectral sequence for $MU^{((G))}$, $d_3(u_{2\sigma})$ and $d_7(u_{2\sigma}^2)$ must be G-invariant since $u_{2\sigma}$ is, and they must map respectively to $a_\sigma^3 \overline{r}_1^{G'}$ and $a_\sigma^7 \overline{r}_3^{G'}$, so we have

$$\begin{aligned}
d_3(u_{2\sigma_2}) &= a_{\sigma_2}^3(\overline{r}_{1,0}^G + \overline{r}_{1,1}^G) \ = \ a_{\sigma_2}^2(\eta_0 + \eta_1) \\
d_7(u_{2\sigma_2}^2) &= a_{\sigma_2}^7 \left(5\overline{r}_{1,0}^G \overline{r}_{1,1}^G(\overline{r}_{1,0}^G + \overline{r}_{1,1}^G) + (\overline{r}_{1,1}^G)^3 + \cdots\right) \\
&= a_{\sigma_2}^7(\overline{r}_{1,0}^G)^3 + \cdots \qquad \text{since } a_{\sigma_2}^3(\overline{r}_{1,0}^G + \overline{r}_{1,1}^G) = 0 \text{ in } \underline{E}_4
\end{aligned}$$

and similarly for $k_{\mathbf{H}}$ where the missing terms in $d_7(u_{2\sigma_2}^2)$ vanish. Pulling back along the isomorphism $\underline{r}_2^4$ and the monomorphism Res_2^4 leads to the following.

Proposition 32. The differentials on u_λ and $u_{2\sigma}$. *The following differentials occur in the slice spectral sequence for $k_{\mathbf{H}}$.*

$$\begin{aligned}
d_3(u_\lambda) &= a_\lambda \eta \\
d_3(\mathrm{Res}_2^4(u_\lambda)) &= \mathrm{Res}_2^4(a_\lambda)(\eta_0 + \eta_1) \\
d_3(u_\sigma \mathrm{Res}_2^4(u_\lambda)) &= u_\sigma \mathrm{Res}_2^4(a_\lambda)(\eta_0 + \eta_1) = \eta_0^2 \eta_1 + \eta_0 \eta_1^2 \\
d_3(u_\lambda \eta) &= a_\lambda \eta^2 = a_\lambda^2 \mathrm{Tr}_2^4(u_\sigma \overline{s}_{2,\epsilon}) \\
d_3(\mathrm{Res}_2^4(u_\lambda)\eta_\epsilon) &= \mathrm{Res}_2^4(a_\lambda)(\eta_0 + \eta_1)\eta_\epsilon \\
d_3(u_\sigma \mathrm{Res}_2^4(u_\lambda)\eta_\epsilon) &= u_\sigma \mathrm{Res}_2^4(a_\lambda)(\eta_0 + \eta_1)\eta_\epsilon \\
&= (\eta_0 \eta_1)^2 + \eta_0 \eta_1 \eta_\epsilon^2 \\
d_5(u_{2\sigma}) &= a_\sigma^3 a_\lambda \bar{\mathfrak{d}}_1 \\
d_5(u_\lambda^2) &= a_\lambda^2 a_\sigma u_\lambda \bar{\mathfrak{d}}_1 = a_\lambda u_\lambda f_1 \\
d_7(\mathrm{Res}_2^4(u_\lambda)^2) &= \mathrm{Res}_2^4(a_\lambda^2)\eta_0^3.
\end{aligned}$$

The elements u_σ, $u_{2\sigma}^2$ and $\mathrm{Res}_2^4(u_\lambda)^2$ are permanent cycles. The first satisfies

$$\mathrm{Tr}_2^4(u_\sigma \mathrm{Res}_2^4(x)) = a_\sigma f_1 x \in \underline{\pi}_{1+|x|} k_{\mathbf{H}}(G/G).$$

Since the slice filtrations of $u_\sigma \mathrm{Res}_2^4(x)$ and its transfer exceed those of x by 0 and 4 respectively, we have an exotic Mackey functor extension.

Proof. The differentials were established above.

Note that $u_\sigma \in \underline{E}_2^{0,1-\sigma}(G/G')$ since the maximal subgroup for which the sign representation σ is oriented is G', on which it is restricts to the trivial representation of degree 1. This group depends only on the restriction of the $RO(G)$-grading to

`Proof needed for` $\mathrm{Tr}_2^4(u_\sigma)$

G', and the isomorphism extends to differentials as well. This means that u_σ is a place holder corresponding to the permanent cycle $1 \in \underline{E}_2^{0,0}(G/G')$.

As remarked above, we lose no information by inverting the class D, which is divisible by $\bar{\mathfrak{d}}_1$. It is shown in [HHR, 9.11] that inverting the latter makes $u_{2\sigma}^2$ a permanent cycle. $\square$

8. $k_{\mathbf{H}}$ as a C_2-spectrum

It is helpful to explore the restriction of the slice spectral sequence to G', for which the $\mathbf{Z}$-brigraded portion $\underline{E}_2(G'/G')$ is the isomorphic image of the ring of Corollary 28. In the following we identify Σ_2, δ_1 and $\overline{r}_1$ with their images under $\underline{r}_2^4$. From the differentials of (31) we get

$$\begin{aligned} d_3(\Sigma_{2,\epsilon}) &= \eta_\epsilon^3 + \eta_\epsilon^2\eta_{1+\epsilon} \\ d_3(\delta_1) &= \eta_0^2\eta_1 + \eta_0\eta_1^2 \\ d_7(\delta_1^2) &= d_7(u_{2\sigma}^2)\overline{r}_{1,0}^2\overline{r}_{1,1}^2 \;=\; a_\sigma^7\overline{r}_3^{G'}\overline{r}_{1,0}^2\overline{r}_{1,1}^2 \\ &= a_\sigma^7(5\overline{r}_1^2\gamma(\overline{r}_1) + 5\overline{r}_{1,0}\overline{r}_{1,1}^2 + \gamma(\overline{r}_1)^3)\overline{r}_{1,0}^2\overline{r}_{1,1}^2. \end{aligned}$$

The d_3s above make all monomials in η_0 and η_1 of any given degree ≥ 3 the same in $\underline{E}_4(G/G')$ and $\underline{E}_4(G'/G')$, so $d_7(\delta_1^2) = \eta_0^7$. Similar calculations show that

$$d_7(\Sigma_{2,\epsilon}^2) = \eta_0^7.$$

This leads to the following, for which Figure 4 is a visual aid.

Theorem 33. The slice spectral sequence for $k_{\mathbf{H}}$ as a C_2-spectrum. *Using the notation of Table 2 and Definition 14, we have*

$$\begin{aligned} \underline{E}_2^{*,*}(G'/e) &= \mathbf{Z}[r_{1,0}, r_{1,1}] \qquad \textit{with } r_{1,\epsilon} \in \underline{E}_2^{0,2}(G'/e) \\ \underline{E}_2^{*,*}(G'/G') &= \mathbf{Z}[\delta_1, \Sigma_{2,\epsilon}, \eta_\epsilon \colon \epsilon = 0,\, 1]/ \\ &\qquad \left(2\eta_\epsilon, \delta_1^2 - \Sigma_{2,0}\Sigma_{2,1}, \eta_\epsilon\Sigma_{2,\epsilon+1} + \eta_{1+\epsilon}\delta_1\right), \end{aligned}$$

so

$$\underline{E}_2^{s,t} = \begin{cases} \square \oplus \bigoplus_\ell \widetilde{\square} & \textit{for } (s,t) = (0, 4\ell) \textit{ with } \ell \geq 0 \\ \bigoplus_{\ell+1} \widetilde{\widetilde{\square}} & \textit{for } (s,t) = (0, 4\ell+2) \textit{ with } \ell \geq 0 \\ \bullet \oplus \bigoplus_{u+\ell} \widetilde{\bullet} & \textit{for } (s,t) = (2u, 4\ell+4u) \textit{ with } \ell \geq 0 \textit{ and } u > 0 \\ \bigoplus_{u+\ell} \widetilde{\bullet} & \textit{for } (s,t) = (2u-1, 4\ell+4u-2) \textit{ with } \ell \geq 0 \textit{ and } u > 0 \\ 0 & \textit{otherwise.} \end{cases}$$

The first differentials are determined by

$$d_3(\Sigma_{2,\epsilon}) = \eta_\epsilon^2(\eta_0 + \eta_1) \qquad \textit{and} \qquad d_3(\delta_1) = \eta_0\eta_1(\eta_0 + \eta_1)$$

resulting in

$$\underline{E}_4^{s,t} = \begin{cases} \square \oplus \bigoplus_\ell \widetilde{\square} & \textit{for } (s,t) = (0,4\ell) \textit{ with } \ell \geq 0 \textit{ and } \ell \textit{ even} \\ \text{◪} \oplus \bigoplus_\ell \widetilde{\text{◪}} & \textit{for } (s,t) = (0,4\ell) \textit{ with } \ell \geq 0 \textit{ and } \ell \textit{ odd} \\ \bigoplus_{\ell+1} \widetilde{\widetilde{\square}} & \textit{for } (s,t) = (0,4\ell+2) \textit{ with } \ell \geq 0 \\ \bullet \oplus \bigoplus_{1+\ell} \widetilde{\bullet} & \textit{for } (s,t) = (2,4\ell+4) \textit{ with } \ell \geq 0 \textit{ even} \\ \bigoplus_{1+\ell} \widetilde{\bullet} & \textit{for } (s,t) = (1,4\ell+2) \textit{ with } \ell \geq 0 \textit{ even} \\ \bullet & \textit{for } (s,t) = (s,4\ell+2s) \textit{ with } s \geq 3 \textit{ and } \ell \geq 0 \textit{ even} \\ 0 & \textit{otherwise.} \end{cases}$$

There is a second set of differentials determined by

$$d_7(\Sigma_{2,\epsilon}) = d_7(\delta_1) = \eta_0^7$$

resulting in

$$\underline{E}_8^{s,t} = \underline{E}_\infty^{s,t} = \begin{cases} \square \oplus \bigoplus_\ell \widetilde{\square} & \textit{for } (s,t) = (0,4\ell) \textit{ with } \ell \geq 0 \textit{ and } \ell \textit{ divisible by } 4 \\ \text{◪} \oplus \bigoplus_\ell \widetilde{\square} & \textit{for } (s,t) = (0,4\ell) \textit{ with } \ell \geq 0 \textit{ and } \ell \equiv 2 \textit{ mod } 4 \\ \text{◪} \oplus \bigoplus_\ell \widetilde{\text{◪}} & \textit{for } (s,t) = (0,4\ell) \textit{ with } \ell \geq 0 \textit{ and } \ell \textit{ odd} \\ \bigoplus_{\ell+1} \widetilde{\widetilde{\square}} & \textit{for } (s,t) = (0,4\ell+2) \textit{ with } \ell \geq 0 \\ \bullet \oplus \bigoplus_{1+\ell} \widetilde{\bullet} & \textit{for } (s,t) = (2,4\ell+4) \textit{ with } \ell \geq 0 \textit{ divisible by } 4 \\ \bigoplus_{1+\ell} \widetilde{\bullet} & \textit{for } (s,t) = (2,4\ell+4) \textit{ with } \ell \geq 0 \textit{ and } \ell \equiv 2 \textit{ mod } 4 \\ \bigoplus_{1+\ell} \widetilde{\bullet} & \textit{for } (s,t) = (1,4\ell+2) \textit{ with } \ell \geq 0 \textit{ divisble by } 4 \\ \bullet \oplus \bigoplus_{\ell} \widetilde{\bullet} & \textit{for } (s,t) = (1,4\ell+2) \textit{ with } \ell \geq 0 \textit{ and } \ell \equiv 2 \textit{ mod } 4 \\ \bullet & \textit{for } (s,t) = (s,4\ell+2s) \textit{ with } 3 \leq s \leq 6 \textit{ and } \ell \geq 0 \\ & \qquad \textit{divisible by } 4 \\ 0 & \textit{otherwise.} \end{cases}$$

Corollary 34. Some nontrivial permanent cycles. *The following elements in* $\underline{E}_2^{s,8i+2s} k_{\mathbf{H}}(G/G')$ *and their transfers are nontrivial permanent cycles:*

- $\Sigma_{2,\epsilon}^{2i-j}\delta_1^j$ *for* $0 \leq j \leq 2i$ *(*$4i+1$ *elements of infinite order including* δ_1^{2i}*),* i *even and* $s = 0$*.*
- $\eta_\epsilon \Sigma_{2,\epsilon}^{2i-j}\delta_1^j$ *for* $0 \leq j < 2i$ *and* $\eta_\epsilon \delta_1^{2i}$ *(*$4i+2$ *elements or order 2) for* i *even and* $s = 1$*.*
- $\eta_\epsilon^2 \Sigma_{2,\epsilon}^{2i-j}\delta_1^j$ *for* $0 \leq j < 2i$ *and* $\delta_1^{2i}\left\{\eta_0^2,\ \eta_0\eta_1,\ \eta_1^2\right\}$ *(*$4i+3$ *elements or order 2) for* i *even and* $s = 2$*.*
- $\eta_0^s \delta_1^{2i}$ *for* $3 \leq s \leq 6$ *(4 elements or order 2) and* i *even.*
- $\Sigma_{2,\epsilon}^{2i-j}\delta_1^j + \delta_1^{2i}$ *for* $0 \leq j \leq 2i$ *(*$4i+1$ *elements of infinite order including* $2\delta_1^{2i}$*),* i *odd and* $s = 0$*.*
- $\eta_\epsilon \Sigma_{2,\epsilon}^{2i-j}\delta_1^j + \delta_1^{2i}$ *for* $0 \leq j \leq 2i-1$ *and* $\eta_0\delta_1^{2i-1}(\Sigma_{2,1}+\delta_1) = \eta_1\delta_1^{2i-1}(\Sigma_{2,0}+\delta_1)$ *(*$4i+1$ *elements of order 2),* i *odd and* $s = 1$*.*
- $\eta_\epsilon^2 \Sigma_{2,\epsilon}^{2i-j}\delta_1^j + \delta_1^{2i}$ *for* $0 \leq j \leq 2i-1$*,* $\eta_0^2\delta_1^{2i-1}(\Sigma_{2,1}+\delta_1) = \eta_0\eta_1\delta_1^{2i-1}(\Sigma_{2,0}+\delta_1)$ *and* $\eta_0\eta_1\delta_1^{2i-1}(\Sigma_{2,1}+\delta_1) = \eta_1^2\delta_1^{2i-1}(\Sigma_{2,0}+\delta_1)$ *(*$4i+2$ *elements of order 2) for* i *odd and* $s = 2$*.*

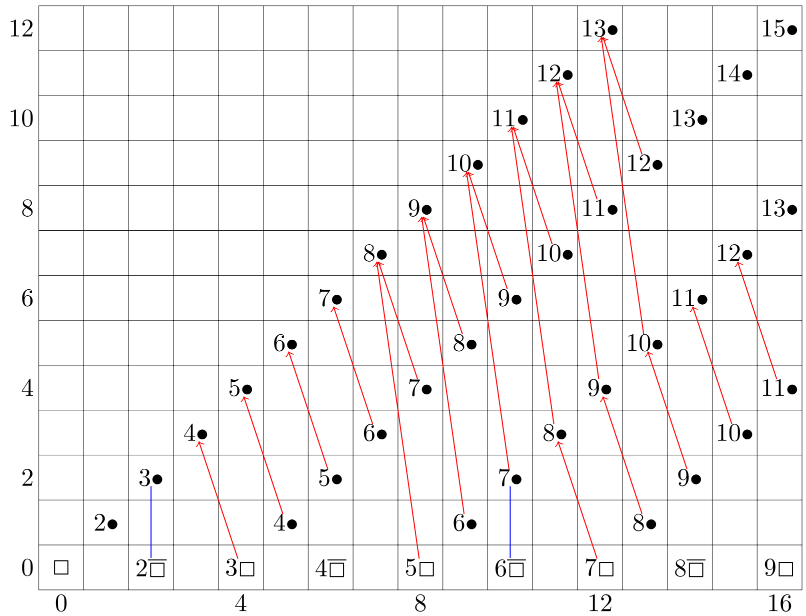


FIGURE 4. The slice spectral sequence for $k_{\mathbf{H}}$ as a C_2-spectrum. The Mackey functor symbols are as in Table 2, The C_4-structure of the Mackey functors is not indicated here. In each bidegree we have a direct sum of the indicated number of the indicated Mackey functor. Each d_3 has maximal rank, leaving a cokernel of rank 1, and each d_7 has rank 1. Blue lines indicate exotic transfers, which also have maximal rank.

In $\underline{E}_2^{0,8i+4} k_{\mathbf{H}}(G/G')$ *we have* $2\Sigma_{2,\epsilon}^{2i+1-j}\delta_1^j$ *for* $0 \leq j \leq 2i$ *and* $2\delta_1^j$, $4i+3$ *elements of infinite order, each in the image of the transfer* Tr_1^2.

9. THE FIRST DIFFERENTIALS OVER C_4

Theorem 27 lists elements in the slice spectral sequence for $k_{\mathbf{H}}$ over C_4 in terms of

$$r_1, \, \bar{s}_2, \, \bar{\mathfrak{d}}_1; \quad \eta, \, a_\sigma, \, a_\lambda; \quad u_\lambda, \, u_\sigma, \text{ and } u_{2\sigma}.$$

All but the u's are permanent cycles, and the action of d_3 on u_λ, u_σ and $u_{2\sigma}$ is described above in Proposition 32.

Proposition 35. **d_3 on elements in Theorem 27.** *We have the following* d_3*s, subject to the conditions on* i, j, k *and* ℓ *of Theorem 27:*

- *On* $X_{2\ell,2\ell}$*:*

$$\begin{aligned} d_3(a_\lambda^j u_{2\sigma}^\ell u_\lambda^{2\ell-j}\bar{\mathfrak{d}}_1^{2\ell}) &= \begin{cases} a_\lambda^{j+1}\eta u_{2\sigma}^\ell u_\lambda^{2\ell-j-1}\bar{\mathfrak{d}}_1^{2\ell} \\ \qquad \in \underline{\pi}_* X_{2\ell,2\ell+1}(G/G) \\ \qquad\qquad \text{for } j \text{ odd} \\ 0 \qquad\qquad \text{for } j \text{ even} \end{cases} \\ d_3(a_\sigma^{2k} a_\lambda^{2\ell} u_{2\sigma}^{\ell-k}\bar{\mathfrak{d}}_1^{2\ell}) &= 0 \end{aligned}$$

- *On* $X_{2\ell+1,2\ell+1}$*:*

$$\begin{aligned} d_3(\delta_1^{2\ell+1}) &= \eta u_\sigma^{2\ell+1}\mathrm{Res}_2^4(a_\lambda u_\lambda^{2\ell}\bar{\mathfrak{d}}_1^{2\ell+1}) \\ &\qquad \in \underline{\pi}_* X_{2\ell+1,2\ell+2}(G/G') \\ d_3(u_\sigma^{2\ell+1}&\mathrm{Res}_2^4(a_\lambda^j u_\lambda^{2\ell+1-j}\bar{\mathfrak{d}}_1^{2\ell+1})) \\ &= \begin{cases} \eta u_\sigma^{2\ell+1}\mathrm{Res}_2^4(a_\lambda^{j+1} u_\lambda^{2\ell-j}\bar{\mathfrak{d}}_1^{2\ell+1}) \\ \qquad \in \underline{\pi}_* X_{2\ell+1,2\ell+2}(G/G') \\ \qquad\qquad \text{for } j \text{ even} \\ 0 \qquad\qquad \text{for } j \text{ odd} \end{cases} \\ d_3(a_\sigma a_\lambda^j u_\sigma^{2\ell} u_\lambda^{2\ell+1-j}\bar{\mathfrak{d}}_1^{2\ell+1}) &= \begin{cases} \eta a_\sigma a_\lambda^{j+1} u_\sigma^{2\ell} u_\lambda^{2\ell-j}\bar{\mathfrak{d}}_1^{2\ell+1} \\ \qquad \in \underline{\pi}_* X_{2\ell+1,2\ell+2}(G/G) \\ \qquad\qquad \text{for } j \text{ even} \\ 0 \qquad\qquad \text{for } j \text{ odd} \end{cases} \\ d_3(a_\sigma^{2k+1} a_\lambda^{2\ell+1} u_{2\sigma}^{\ell-k}\bar{\mathfrak{d}}_1^{2\ell+1}) &= 0 \end{aligned}$$

- *On* $X_{i,2\ell-i}$*:*

$$\begin{aligned} d_3(u_\sigma^\ell \bar{s}_2^{\ell-i}\mathrm{Res}_2^4(u_\lambda^\ell \bar{\mathfrak{d}}_1^i)) &= \begin{cases} \eta^3 u_\sigma^{\ell-1}\bar{s}_2^{\ell-i-1}\mathrm{Res}_2^4(u_\lambda^{\ell-1}\bar{\mathfrak{d}}_1^i) \\ \qquad \in \underline{\pi}_* X_{i,2\ell+1-i}(G/G') \\ \qquad\qquad \text{for } \ell \text{ odd} \\ 0 \qquad\qquad \text{for } \ell \text{ even} \end{cases} \\ d_3(\eta^{2j} u_\sigma^{\ell-j}\bar{s}_2^{\ell-i-j}\mathrm{Res}_2^4(u_\lambda^{\ell-j}\bar{\mathfrak{d}}_1^i)) &= \begin{cases} \eta^{2j+1} u_\sigma^{\ell-j}\bar{s}_2^{\ell-i-j}\mathrm{Res}_2^4(a_\lambda u_\lambda^{\ell-j-1}\bar{\mathfrak{d}}_1^i) \\ \qquad \in \underline{\pi}_* X_{i,2\ell+1-i}(G/G') \\ \qquad\qquad \text{for } \ell-j \text{ odd} \\ 0 \qquad\qquad \text{for } \ell-j \text{ even} \end{cases} \end{aligned}$$

- *On* $X_{i,2\ell+1-i}$*:*

$$\begin{aligned} d_3(r_1\mathrm{Res}_1^2(u_\sigma^\ell \bar{s}_2^{\ell-i})\mathrm{Res}_1^4(u_\lambda^\ell \bar{\mathfrak{d}}_1^i)) &= 0 \\ d_3(\eta^{2j+1} u_\sigma^{\ell-j}\bar{s}_2^{\ell-i-j}\mathrm{Res}_2^4(u_\lambda^{\ell-j}\bar{\mathfrak{d}}_1^i)) &= \begin{cases} \eta^{2j+2} u_\sigma^{\ell-j}\bar{s}_2^{\ell-i-j}\mathrm{Res}_2^4(a_\lambda u_\lambda^{\ell-j-1}\bar{\mathfrak{d}}_1^i) \\ \qquad \in \underline{\pi}_* X_{i,2\ell+2-i}(G/G') \\ \qquad\qquad \text{for } \ell-j \text{ odd} \\ 0 \qquad\qquad \text{for } \ell-j \text{ even} \end{cases} \end{aligned}$$

Note that in each case the first index of X is unchanged by the differential, and the second one is increased by one. Since $X_{m,n}$ is a summand of the $2(m+n)$th slice, each d_3 raises the slice degree by 2 as expected.

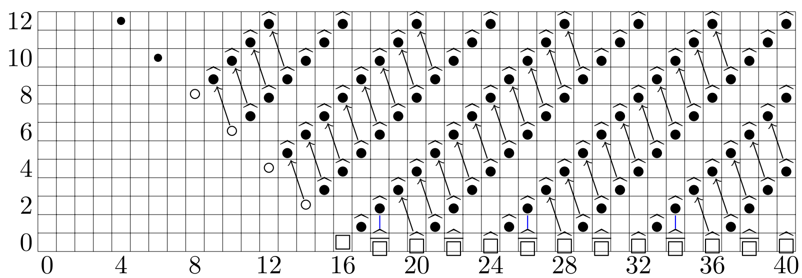


FIGURE 5. The d_3s on the slice summands $X_{4,n}$ for $n \geq 4$. The symbols are defined in Table 1.

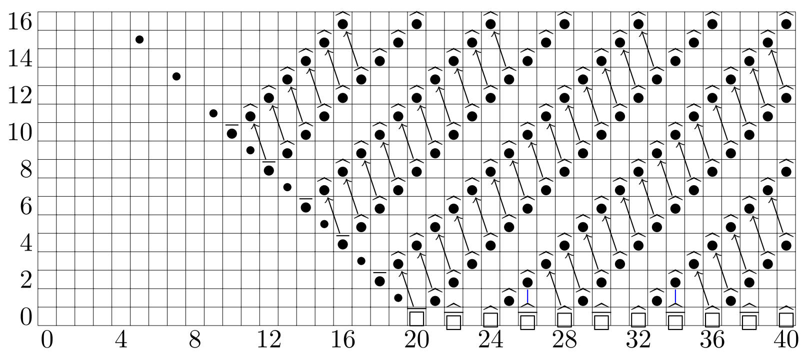
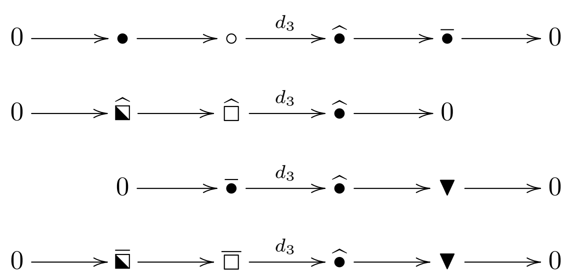


FIGURE 6. The d_3s on the slice summands $X_{5,n}$ for $n \geq 5$.

These differentials are illustrated in Figures 5 and 6. In order to pass to $\underline{E}_4$ we need the following exact sequences of Mackey functors.



The resulting subquotients of $\underline{E}_4$ are shown in Figures 7 and 8 and described below in Theorem 36. In the latter the slice summands are organized as shown in the Figures rather than by orbit type as in Theorem 27.

Theorem 36. The slice $\underline{E}_4$-term for $k_{\mathbf{H}}$. *The elements of Theorem 27 surviving to $\underline{E}_4$, which live in the appropriate subquotients of $\underline{\pi}_* X_{m,n}$, are as follows.*

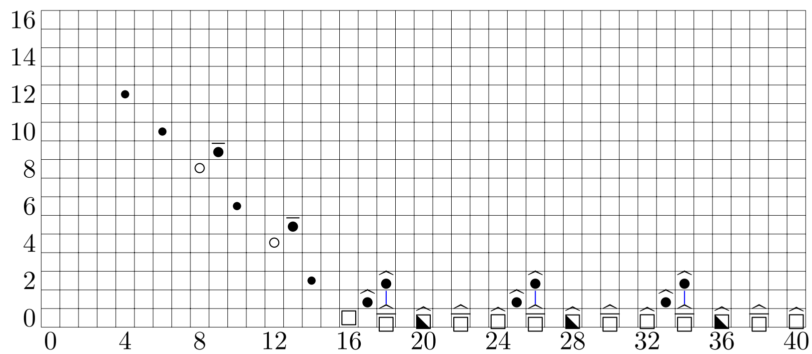


FIGURE 7. The subquotient of the slice $\underline{E}_4$-term for $k_{\mathbf{H}}$ for the slice summands $X_{4,n}$ for $n \geq 4$. Exotic transfers are shown in blue.

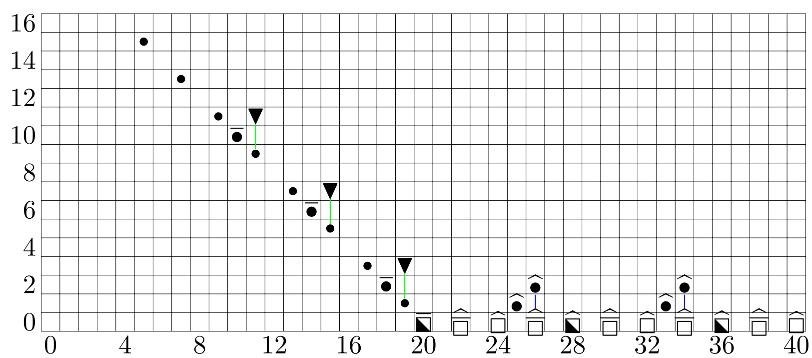


FIGURE 8. The subquotient of the slice $\underline{E}_4$-term for $k_{\mathbf{H}}$ for the slice summands $X_{5,n}$ for $n \geq 5$. Exotic restrictions and transfers are shown in green and blue respectively.

(1) *In $\underline{\pi}_* X_{2\ell,2\ell}$ (see the leftmost diagonal in Figure 7), on the 0-line we still have a copy of $\square$ generated under fixed point restrictions by $\Delta_1^\ell \in \underline{E}_4^{0,8\ell}$. In positive filtrations we have*

$$\begin{aligned}
\circ &\subseteq \underline{E}_4^{2j,8\ell} && \text{generated by} \\
a_\lambda^j u_{2\sigma}^\ell u_\lambda^{2\ell-j} \bar{\mathfrak{d}}_1^{2\ell} &\in \underline{E}_4^{2j,8\ell}(G/G) && \text{for } j \text{ even and } 0 < j \leq 2\ell, \\
2a_\lambda^j u_{2\sigma}^\ell u_\lambda^{2\ell-j} \bar{\mathfrak{d}}_1^{2\ell} &= a_\sigma^2 a_\lambda^{j-1} u_{2\sigma}^{\ell+1} u_\lambda^{2\ell-j-1} \bar{\mathfrak{d}}_1^{2\ell} \\
&\in \underline{E}_4^{2j,8\ell}(G/G) && \text{for } j \text{ odd and } 0 < j \leq 2\ell \text{ and} \\
\bullet &\subseteq \underline{E}_4^{2k+2\ell,8\ell} && \text{generated by} \\
a_\sigma^{2k} a_\lambda^{2\ell} u_{2\sigma}^{\ell-k} \bar{\mathfrak{d}}_1^{2\ell} &\in \underline{E}_4^{2j+2k,8\ell}(G/G) && \text{for } 0 < k \leq \ell.
\end{aligned}$$

(2) *In $\underline{\pi}_* X_{2\ell,2\ell+1}$ (see the second leftmost diagonal in Figure 7), in filtration 0 we have $\widehat{\widehat{\square}}$, generated (under transfers and the group action) by*

$$r_1 \mathrm{Res}_1^2(u_\sigma^{2\ell} \mathrm{Res}_1^4(u_\lambda^{2\ell} \bar{\mathfrak{d}}_1^{2\ell}) \in \underline{E}_2^{0,8\ell+2}(G/e).$$

In positive filtrations we have

$$\begin{aligned} \widehat{\bullet} &\subseteq \underline{E}_4^{1,8\ell+2} && \textit{generated (under transfers and the group action) by} \\ \eta u_\sigma^{2\ell} \mathrm{Res}_2^4(u_\lambda \bar{\mathfrak{d}}_1)^{2\ell} &= \underline{E}_4^{1,8\ell+2}(G/G') \\ \overline{\bullet} &\subseteq \underline{E}_4^{4k+1,8\ell+2} && \textit{for } 0 < k \leq \ell \textit{ generated by} \\ x = \eta^{4k+1} u_\sigma^{2\ell-2k} &\mathrm{Res}_2^4(u_\lambda \bar{\mathfrak{d}}_1)^{2\ell-2k} \\ &\in \underline{E}_4^{4k+1,8\ell+2}(G/G') && \textit{with } (1-\gamma)x = \mathrm{Tr}_2^4(x) = 0. \end{aligned}$$

(3) *In $\underline{\pi}_* X_{2\ell+1,2\ell+1}$ (see the leftmost diagonal in Figure 8), on the 0-line we have a copy of $\overline{\blacksquare}$ generated under fixed point $\Delta_1^{(2\ell+1)/2} \in \underline{E}_4^{0,8\ell+4}$. In positive filtrations we have*

$$\begin{aligned} \overline{\bullet} &\subseteq \underline{E}_2^{2j,8\ell+4} && \textit{generated by} \\ u_\sigma^{2\ell+1} \mathrm{Res}_2^4(a_\lambda^j & u_\lambda^{2\ell+1-j} \bar{\mathfrak{d}}_1^{2\ell+1}) \\ &\in \underline{E}_2^{2j,8\ell+4}(G/G') && \textit{for } 0 < j \leq 2\ell+1, \\ \bullet &\subseteq \underline{E}_2^{2j+1,8\ell+4} && \textit{generated by} \\ a_\sigma a_\lambda^j u_\sigma^{2\ell} u_\lambda^{2\ell+1-j} \bar{\mathfrak{d}}_1^{2\ell+1} &\in \underline{E}_2^{2j+2k,8\ell+4}(G/G) && \textit{for } 0 \leq j \leq 2\ell+1 \textit{ and} \\ \bullet &\subseteq \underline{E}_2^{2k+4\ell+3,8\ell+4} && \textit{generated by} \\ a_\sigma^{2k+1} a_\lambda^{2\ell+1} u_{2\sigma}^{\ell-k} \bar{\mathfrak{d}}_1^{2\ell+1} &\in \underline{E}_2^{2k+4\ell+2,8\ell+4}(G/G) && \textit{for } 0 < k \leq 2\ell+1. \end{aligned}$$

(4) *In $\underline{\pi}_* X_{2\ell+1,2\ell+2}$ (see the second leftmost diagonal in Figure 8), in filtration 0 we have $\widehat{\widehat{\square}}$, generated (under transfers and the group action) by*

$$r_1 \mathrm{Res}_1^2(u_\sigma^{2\ell+1} \mathrm{Res}_1^4(u_\lambda^{2\ell+1} \bar{\mathfrak{d}}_1^{2\ell+1}) \in \underline{E}_4^{0,8\ell+6}(G/e).$$

In positive filtrations we have

$$\begin{aligned} \blacktriangledown &\subseteq \underline{E}_4^{04k+3,8\ell+6} && \textit{for } 0 \leq k \leq \ell \\ & && \textit{generated under transfer by} \\ x = \eta^{4k+3} \Delta_1^{\ell-k} &\in \underline{E}_4^{4k+3,8\ell+6}(G/G') && \textit{with } (1-\gamma)x = 0. \end{aligned}$$

(5) *In $\underline{\pi}_* X_{m,m+i}$ for $i \geq 2$ (see the rest of Figures 7 and 8), in filtration 0 we have*

$$\begin{aligned} \widehat{\widehat{\square}} &\subseteq \underline{E}_4^{0,4m+4j+2} && \textit{generated under transfers and group action by} \\ r_1 \mathrm{Res}_1^2(u_\sigma^{m+j} \bar{s}_2^j) & \mathrm{Res}_1^4(u_\lambda^{m+j} \bar{\mathfrak{d}}_1^m) \\ &\in \underline{E}_4^{0,4m+4j+2}(G/e) && \textit{for } j \geq 0 \\ \widehat{\overline{\blacksquare}} &\subseteq \underline{E}_4^{0,8\ell+4} && \textit{generated under transfers and group action by} \\ \mathrm{Res}_1^2(u_\sigma^{2\ell+1} \bar{s}_2^{2\ell+1-m}) & \mathrm{Res}_1^4(u_\lambda^{2\ell+1} \bar{\mathfrak{d}}_1^m) \end{aligned}$$

$$
\begin{array}{rcl}
 & \in & \underline{E}_4^{0,8\ell+4}(G/e) \quad \textit{for } \ell \geq m/2 \\
\widehat{\Box} & \subseteq & \underline{E}_4^{0,8\ell} \quad \textit{generated under transfers and restriction} \\
 & & \quad \textit{and group action by} \\
x_{8\ell,m} & = & \Sigma_{2,0}^{2\ell-m}\delta_1^m + \ell\delta_1^{2\ell} \\
 & & \quad \textit{where } \Sigma_{2,\epsilon} = u_{\rho_2}\overline{s}_{2,\epsilon} \textit{ and } \delta_1 = u_{\rho_2}\mathrm{Res}_2^4(\bar{\mathfrak{d}}_1) \\
 & \in & \underline{E}_4^{0,8\ell}(G/G') \quad \textit{for } 0 \leq m \leq 2\ell - 1.
\end{array}
$$

In positive filtrations we have

$$
\begin{array}{rcl}
\widehat{\bullet} & \subseteq & \underline{E}_4^{2,8\ell+4} \quad \textit{generated under transfers} \\
 & & \quad \textit{and group action by} \\
\eta_0^2\mathrm{Res}_2^4(\Delta_1^\ell) & = & \eta_0^2\delta_1^{2\ell} = \eta_0^2 u_\sigma^{2\ell}\mathrm{Res}_2^4(u_\lambda\bar{\mathfrak{d}}_1)^{2\ell} \\
 & \in & \underline{E}_4^{2,8\ell+4}(G/G') \quad \textit{and} \\
\widehat{\bullet} & \subseteq & \underline{E}_4^{s,8\ell+2s} \quad \textit{generated under transfers} \\
 & & \quad \textit{and group action by} \\
\eta_\epsilon^s x_{8\ell,m} & \in & \underline{E}_4^{s,8\ell+2s}(G/G') \qquad \textit{for } s = 1,2 \textit{ and } 0 \leq m \leq 2\ell - 1.
\end{array}
$$

Proposition 37. Some nontrivial permanent cycles. *The elements listed in Theorem 36(5) other than $\eta_\epsilon^2\delta_1^{2\ell}$ are all nontrivial permanent cycles.*

Proof. Each such element is either in the image of $\underline{E}_4^{0,*}(G/e)$ under the transfer and therefore a nontrivial permanent cyle, or it is one of the ones listed in Corollary 34. □

In subsequent discussions and charts, starting with Figure 13, we will omit the elements Proposition 37.

Analogous statements can be made about the slice spectral sequence for $K_{\mathbf{H}}$. Each of its slices is a certain infinite wedge spelled out in Corollary 29. Their homotopy groups are determined by the chain complex calculations of Section 4 and illustrated in Figures 2 and 3. Analogs of Figures 5–8 are shown in Figures 9–12. In each figure, exotic transfers and restrictions are indicated by blue and green lines respsctively. As in the $k_{\mathbf{H}}$ case, most of the elements shown in this chart can be ignored for the purpose of calculating higher differentials.

The resulting reduced $\underline{E}_4$ for $K_{\mathbf{H}}$ is shown in Figure 14. The information shown there is very useful for computing differentials and extensions. The periodicity theorem tells us that $\underline{\pi}_n K_{\mathbf{H}}$ and $\underline{\pi}_{n-32}K_{\mathbf{H}}$ are isomorphic. For $0 \leq n < 32$ these groups appear in the first and third quadrants respectively, and the information visible in the spectral sequence can be quite different.

For example, we see that $\underline{\pi}_0 K_{\mathbf{H}} = \Box$ while $\underline{\pi}_{-32}K_{\mathbf{H}}$ has a subgroup isomorphic to ⬕. The quotient □/⬕ is isomorphic to $\circ$. This leads to the exotic restrictions and transfer in dimension -32 shown in Figure 14. Information that is transparent in dimension 0 implies subtle information in dimension -32. Conversely, we see easily that $\underline{\pi}_{-4}K_{\mathbf{H}} = \dot{\Box}$ while $\underline{\pi}_{28}K_{\mathbf{H}}$ has a quotient isomorphic to $\overline{\text{⬕}}$. This leads to the "long transfer" in dimension 28.

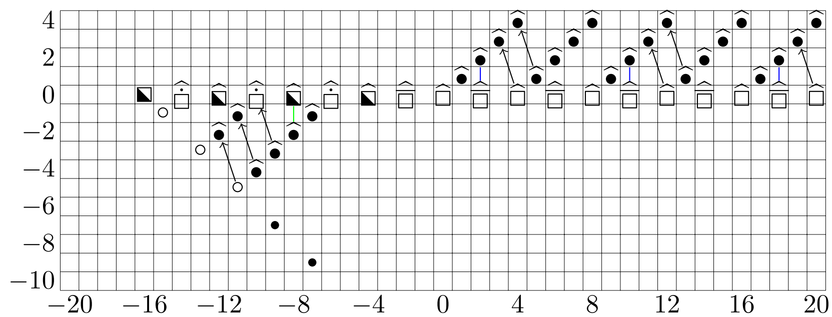


FIGURE 9. The d_3s on the slices $X_{-4,n}$ for $n \geq -4$.

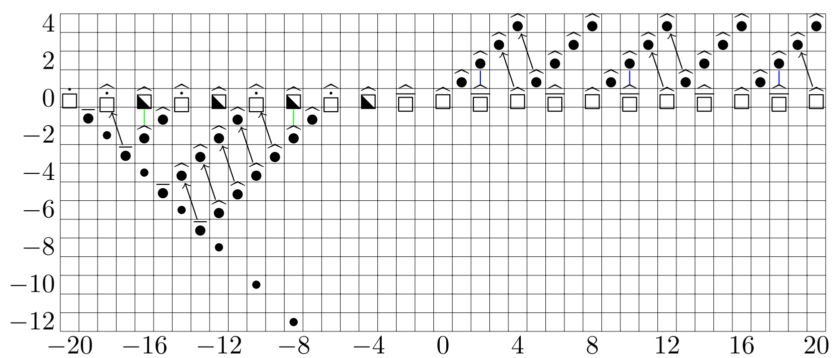


FIGURE 10. The d_3s on the slices $X_{-5,n}$ for $n \geq -5$.

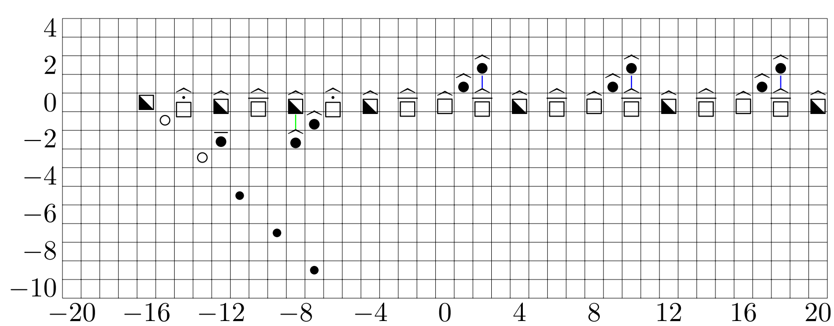


FIGURE 11. The subquotient of the slice $\underline{E}_4$-term for $k_{\mathbf{H}}$ for the slice summands $X_{-4,n}$ for $n \geq -4$.

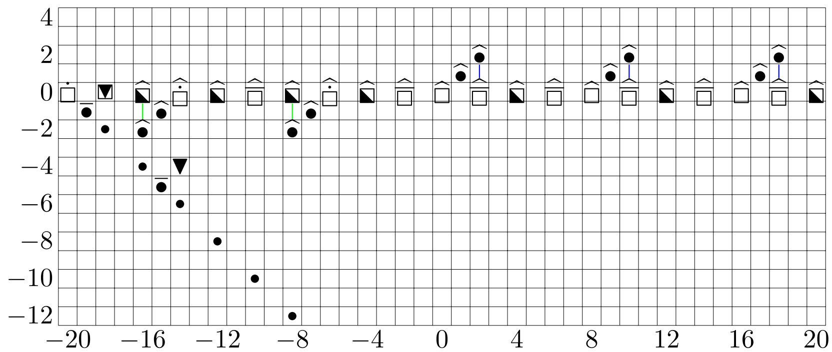


FIGURE 12. The subquotient of the slice $\underline{E}_4$-term for $k_{\mathbf{H}}$ for the slice summands $X_{-5,n}$ for $n \geq -5$.

This technique will be used repeatedly in the proof of Theorem 40 below.

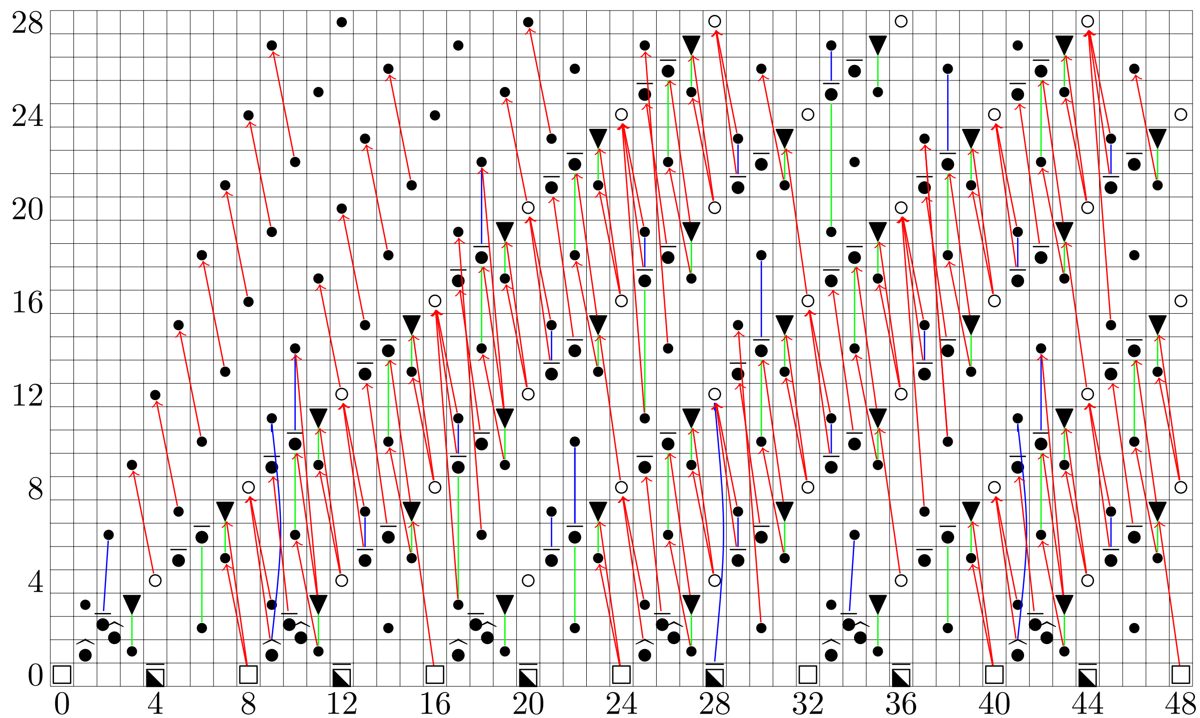


FIGURE 13. The $\underline{E}_4$-term of the slice spectral sequence for $k_{\mathbf{H}}$ with elements of Proposition 37 removed. Differentials are shown in red. Exotic transfers and restrictions are shown in blue and green respectively.

10. Higher differentials and exotic Mackey functor extensions

Theorem 38. The d_5s, d_7s and d_{11}s in the slice spectral sequence for $k_{\mathbf{H}}$. *The $\underline{E}_4$-term of the slice spectral sequence for $k_{\mathbf{H}}$ with elements of Proposition 37 removed (shown in Figure 13) we have*

$$\begin{array}{rcll}
\eta_\epsilon & \in & \underline{E}_4^{1,2}(G/G') & \text{for } \epsilon=0,1 \text{ with } \mathrm{Tr}_2^4(\eta_\epsilon)=\eta\in\underline{E}_4^{1,4}(G/G) \\
f_1 & \in & \underline{E}_4^{3,4}(G/G) & \\
\eta_\epsilon^2,\eta_0\eta_1 & \in & \underline{E}_4^{2,4}(G/G') & \text{with } \mathrm{Tr}_2^4(\eta_\epsilon^2)=\eta^2 \text{ and } \mathrm{Tr}_2^4(\eta_0\eta_1)=f_1^2 \\
\eta_0^3=\eta_0^2\eta_1=\eta_0\eta_1^2=\eta_1^3 & & & \\
 & \in & \underline{E}_4^{3,6}(G/G') & \text{with } \mathrm{Tr}_2^4(\eta_0^3)=x_3\in\underline{E}_4^{3,6}(G/G) \text{ and } \eta^3=0 \\
\nu & \in & \underline{E}_4^{1,4}(G/G) & \text{with } Res_2^4(\nu)=\eta_0^3,\ 2\nu=x_3 \text{ and } Res_2^4(\nu^2)=\eta_0^6 \\
x_4 & \in & \underline{E}_4^{4,8}(G/G) & \text{with } Res_2^4(x_4)=\eta_0^4 \\
y_4=2\delta_1=\mathrm{Tr}_2^4(r_{1,0}r_{1,1}) & & & \\
 & \in & \underline{E}_4^{0,4}(G/G') & \text{with } Tr_2^4(y_4)=0 \\
\Delta_1 & \in & \underline{E}_4^{0,8}(G/G); &
\end{array}$$

see Tables 3 and 4 for more information. All are torsion free under multiplication by x_4 or its restriction η_0^4, except

$$\eta_0+\eta_1,\ \eta_\epsilon^2+\eta_0\eta_1,\quad y_4 \quad \text{and} \quad 4\Delta_1,$$

which are killed by it. Thus the following are linearly independent up to 2-torsion:

$$\begin{array}{c}
\left\{x_4^i\left\{x_3,\ \Delta_1^j,\ f_1\Delta_1^j,\ \nu\Delta_1^j,\ \nu^2\Delta_1^j,\ f_1^{j+2}\colon j\geq 0\right\}:i\geq 0\right\} \\
\cup\left\{\eta_0^{4i}\left\{\eta_0,\ \eta_0^2\right\}:i\geq 0\right\} \\
\cup\left\{\mathrm{Res}_2^4(\Delta_1)^i\left\{y_4,\ \eta_\epsilon,\ \eta_\epsilon^2\right\}:i\geq 0\right\}.
\end{array}$$

There are multiplicative relations

$$f_1\nu=2x_4,\ \nu^2x_4=f_1^2\Delta_1,\ \nu^3=0 \quad \text{and} \quad y_4\eta_\epsilon=0.$$

There are differentials

$$\begin{array}{rcll}
d_5(x_4^{2i+1}f_1^j) & = & x_4^{2i}f_1^{j+3} & \text{for } i,j\geq 0 \\
d_5(\Delta_1^{2i+1}x_4^j) & = & \Delta_1^{2i}\nu x_4^{j+1}=\Delta_1^{2i}x_4^j\langle\nu,\ f_1^2,\ f_1\rangle & \\
d_5(f_1\Delta_1^{2i+1}x_4^j) & = & 2\Delta_1^{2i}x_4^{2_j} & \\
d_7(2\Delta_1^{2i+1}x_4^j) & = & \Delta_1^{2i}x_3x_4^{j+1} & \\
d_7(\Delta_1^{4i+2}x_4^j) & = & j\Delta_1^{4i+1}x_3x_4^{j+1} & \\
d_{11}(\Delta_1^{4i+1}x_3x_4^{2j}) & = & \Delta_1^{4i}f_1^2x_4^{2j+2} &
\end{array}$$

Similar statements can be made about the third quadrant of the slice spectral sequence for $K_{\mathbf{H}}$. They are indicated in Figure 14.

Proof. The structure of the reduced (meaning the elements of Proposition 37 are removed) $\underline{E}_4$-term can be read off from previous calculations.

We have

$$\begin{array}{rcll}
2x_4 & = & 2a_\lambda^2u_{2\sigma}\bar{\mathfrak{d}}_1^2 & \\
 & = & a_{2\sigma}a_\lambda u_\lambda\bar{\mathfrak{d}}_1^2 & \text{since } a_\sigma^2u_\lambda=2a_\lambda u_{2\sigma} \\
d_5(x_4) & = & a_\lambda^2\bar{\mathfrak{d}}_1^2d_5(u_{2\sigma}) & \\
 & = & a_\lambda^2\bar{\mathfrak{d}}_1^2a_\sigma^3a_\lambda\bar{\mathfrak{d}}_1 & \text{by Proposition 32} \\
 & = & f_1^3 &
\end{array}$$

$$\begin{aligned}
d_5(\Delta_1) &= d_5(u_\lambda^2)u_{2\sigma}\bar{\mathfrak{d}}_1^2 + u_\lambda^2 d_5(u_{2\sigma})\bar{\mathfrak{d}}_1^2 \\
&= a_\lambda u_\lambda f_1 u_{2\sigma}\bar{\mathfrak{d}}_1^2 + a_\sigma^3 a_\lambda u_\lambda^2 \bar{\mathfrak{d}}_1^3 \\
&= a_\lambda u_\lambda f_1 u_{2\sigma}\bar{\mathfrak{d}}_1^2 + 2a_\sigma a_\lambda^2 u_\lambda u_{2\sigma}\bar{\mathfrak{d}}_1^3 \qquad \text{since } a_\sigma^2 u_\lambda = 2a_\lambda u_{2\sigma} \\
&= a_\lambda u_\lambda f_1 u_{2\sigma}\bar{\mathfrak{d}}_1^2 \qquad \text{since } 2a_\sigma = 0 \\
&= \nu x_4 = \langle \nu,\, f_1^2,\, f_1\rangle \\
d_5(f_1\Delta_1) &= a_\lambda u_\lambda f_1^2 u_{2\sigma}\bar{\mathfrak{d}}_1^2 \\
&= a_\sigma^2 a_\lambda^3 u_\lambda u_{2\sigma}\bar{\mathfrak{d}}_1^4 \\
&= 2x_4^2 \\
d_7(2\Delta_1) &= d_7(\mathrm{Tr}_2^4(\mathrm{Res}_2^4(\Delta_1))) = \mathrm{Tr}_2^4(d_7(\delta_1^2)) \\
&= \mathrm{Tr}_2^4(\eta_0^7) \qquad \text{by Theorem 33}
\end{aligned}$$

In order to evaluate this transfer note that $\eta_0^4 = \mathrm{Res}_2^4(x_4)$, so

$$\mathrm{Tr}_2^4(\eta_0^7) = \mathrm{Tr}_2^4(\eta_0^3 \mathrm{Res}_2^4(x_4)) = \mathrm{Tr}_2^4(\eta_0^3)x_4 = x_3x_4$$

and $d_7(2\Delta_1) = x_3x_4$.

Since $\mathrm{Res}_2^4(\Delta_1) = \delta_1^2$, we have

$$\begin{aligned}
\mathrm{Res}_2^4(\nu x_4) &= \mathrm{Res}_2^4(\nu)\eta_0^4 \\
&= \mathrm{Res}_2^4(d_5(\Delta_1)) \\
&= d_r(\mathrm{Res}_2^4(\Delta_1)) \qquad \text{for suitable } r \\
&= d_r(\delta_1^2) \\
&= \eta_0^7 \qquad \text{for } r = 7.
\end{aligned}$$

It follows that

$$\eta_0^4(\mathrm{Res}_2^4(\nu) - \eta_0^3) = 0.$$

Since multiplication by η_0^4 maps $\underline{E}_4^{3,6}(G/G')$ isomorphically to $\underline{E}_4^{7,14}(G/G')$, we conclude that

$$\mathrm{Res}_2^4(\nu) = \eta_0^3, \tag{39}$$

which implies that $2\nu = x_3$ and $\mathrm{Res}_2^4(\nu^2) = \eta_0^6$.

Now consider the differential on Δ_1^2. For suitable r we have

$$\begin{aligned}
d_r(\Delta_1^2) &= 2\Delta_1 d_5(\Delta_1) \\
&= 2\Delta_1 \nu x_4 \\
&= \Delta_1 x_3 x_4
\end{aligned}$$

so $d_7(\Delta_1^2) = \Delta_1 x_3 x_4$.

Heuristically we have

Is this rigorous enough?

$$\begin{aligned}
d(\Delta_1^2 x_4) &= d(\Delta_1^2)x_4 + \Delta_1^2 d(x_4) \\
&= \Delta_1 x_3 x_4^2 + \Delta_1^2 f_1^3 \\
&= \Delta_1 x_3 x_4^2 + \Delta_1 f_1(\Delta_1 f_1^2) \\
&= \Delta_1 x_3 x_4^2 + \Delta_1 f_1(\nu^2 x_4) \\
&= \Delta_1 x_3 x_4^2 + \Delta_1 (f_1\nu)\nu x_4 \\
&= \Delta_1 x_3 x_4^2 + \Delta_1 (2x_4)\nu x_4
\end{aligned}$$

$$\begin{array}{rcl} & = & 2\Delta_1 x_3 x_4^2 \\ & = & 0. \end{array}$$

For the d_{11} we have

$$\begin{array}{rcl} d_r(\Delta_1 x_3) & = & d_r(\mathrm{Tr}_2^4(\mathrm{Res}_2^4(\Delta_1\nu))) \\ & = & \mathrm{Tr}_2^4(\mathrm{Res}_2^4(d_5(\Delta_1\nu))) \\ & = & \mathrm{Tr}_2^4(\mathrm{Res}_2^4(\nu^2 x_4)) \\ & = & \mathrm{Tr}_2^4(\eta_0^{10}) \\ & = & \mathrm{Tr}_2^4(\eta_0\eta_1\mathrm{Res}_2^4(x_4^2)) \\ & = & \mathrm{Tr}_2^4(\eta_0\eta_1)x_4^2 \\ & = & f_1^2 x_4^2 \end{array}$$

All other d_5s and d_7s are formal consequences of the above. □

Theorem 40. Higher differentials and Mackey functor extensions in the slice spectral sequence for $k_{\mathbf{H}}$. *In addition to the differentials and Mackey functor extensions of Theorem 38 we have*

$$\begin{array}{lrcl} \textit{in dimension 9:} & \mathrm{Tr}_2^4(\delta_1^{8i+2}(\eta_0+\eta_1)) & = & f_1\Delta_1^{4i}x_4^2 \\ \textit{in dimension 13:} & \mathrm{Tr}_2^4(\delta_1^{4i+2}\eta_0^5 x_4^j) & = & f_1\Delta_1^{2i}x_4^{j+1} \\ \textit{in dimension 17:} & \mathrm{Res}_2^4(f_1\Delta_1^{4i+2}x_4^{2j}) & = & \delta_1^{8i+2}\eta_0^{9+8j} \\ \textit{in dimension 17:} & d_{13}(f_1\Delta_1^{4i+1}x_4^{2j}) & = & \Delta_1^{4i}x_4^{2j+4} \\ \textit{in dimension 18:} & d_{13}(f_1^2\Delta_1^{4i+1}x_4^{2j}) & = & f_1 x_4^4\Delta_1^{4i}x_4^{2j} \\ \textit{in dimension 21:} & \mathrm{Tr}_2^4(\delta_1^{8i+4}\eta_0^5 x_4^j) & = & f_1\Delta_1^{4i+2}x_4^{j+1} \\ \textit{in dimension 22:} & \mathrm{Tr}_2^4(\eta_0^6\delta_1^{8i+4}) & = & f_1^2 x_4\Delta_1^{4i+2} \\ \textit{in dimension 28:} & \mathrm{Tr}_2^4(\delta_1^{8i+7}) & = & x_4^3\Delta_1^{4i+2} \\ \textit{in dimension 38:} & d_{13}(\nu^2\Delta_1^{4i+3}x_4^{2j}) & = & f_1\Delta_1^{4i+2}x_4^{2j+3} \end{array}$$

where $i, j \geq 0$ and the indicated dimension is the one where the phenomenon first occurs, namely the one for $i = j = 0$.

Δ_1^4 is a permanent cycle and there are no other differentials or exotic Mackey functor extensions.

Proof. In dimensions 8 and 9 we have exact sequences

$$\begin{array}{ccccccccc} \underline{E}_7^{3,12} & \longrightarrow & \underline{E}_5^{3,12} & \xrightarrow{d_5} & \underline{E}_5^{8,16} & \longrightarrow & \underline{E}_7^{8,16} & \longrightarrow & 0 \\ \| & & \| & & \| & & \| & & \\ 0 & & \bullet & & \circ & & \blacktriangle & & \end{array}$$

and

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \underline{E}_9^{3,12} & \longrightarrow & \underline{E}_7^{1,10} & \xrightarrow{d_7} & \underline{E}_7^{8,16} & \longrightarrow & \underline{E}_9^{8,16} & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \| & & \\ & & \blacktriangle & & \widehat{\bullet} & & \blacktriangle & & \bullet & & \end{array}$$

From Figure 14 we see that $\underline{\pi}_{-23}K_{\mathbf{H}} = \circ$, so the same must be true of $\underline{\pi}_9 K_{\mathbf{H}}$. This implies the exotic transfer.

In dimension 13 the exotic transfer is required to give a differential leading to $\underline{\pi}_{13}K_{\mathbf{H}} = \underline{\pi}_{-19}K_{\mathbf{H}} = 0$.

The two statements in dimension 17 are equivalent since

$$d_7(\delta_1^2\eta_0^9) = \mathrm{Res}_2^4(x_4^4).$$

The target of the long differential is the square of $\epsilon \in \underline{\pi}_8 K_{\mathbf{H}}(G/G)$, which has filtration 8. The corresponding element in $\underline{\pi}_{-24}$ has filtration -4, so its product with ϵ would have filtration 4 and is therefore 0. Hence ϵ^2 must be hit by a differential, and this d_{13} is the only possibility.

The element $x_4\Delta_1^2$ in dimension 20 is the image of $\overline{\kappa} \in \pi_{20}S^0$. It product with the exotic transfer in dimension 2 gives the one in dimension 22.

`Proofs needed for the rest`

The Periodicity Theorem of [HHR, 9.16] says that Δ_1^4 is a permanent cycle inducing an isomorphism in homotopy. The absence of higher differentials and extensions can be established by careful inspection illustrated in Corollary 41.

Some of the other statements can be proved by comparison with third quadrant calculations in the slice spectral sequence for $K_{\mathbf{H}}$ which we have not described yet, but which is illustrated in Figure 14.

In dimension 9, we find that $\underline{\pi}_{-23}K_{\mathbf{H}} = \underline{E}_\infty^{-1,-24} = \circ$, so $\underline{\pi}_9$ must have the same value.

In dimension 13 the indicated exotic transfer is need to get a torsion free $\underline{\pi}_{12}$. In the dimension -20 we need a d_{13} to achieve the same result. ?????

□

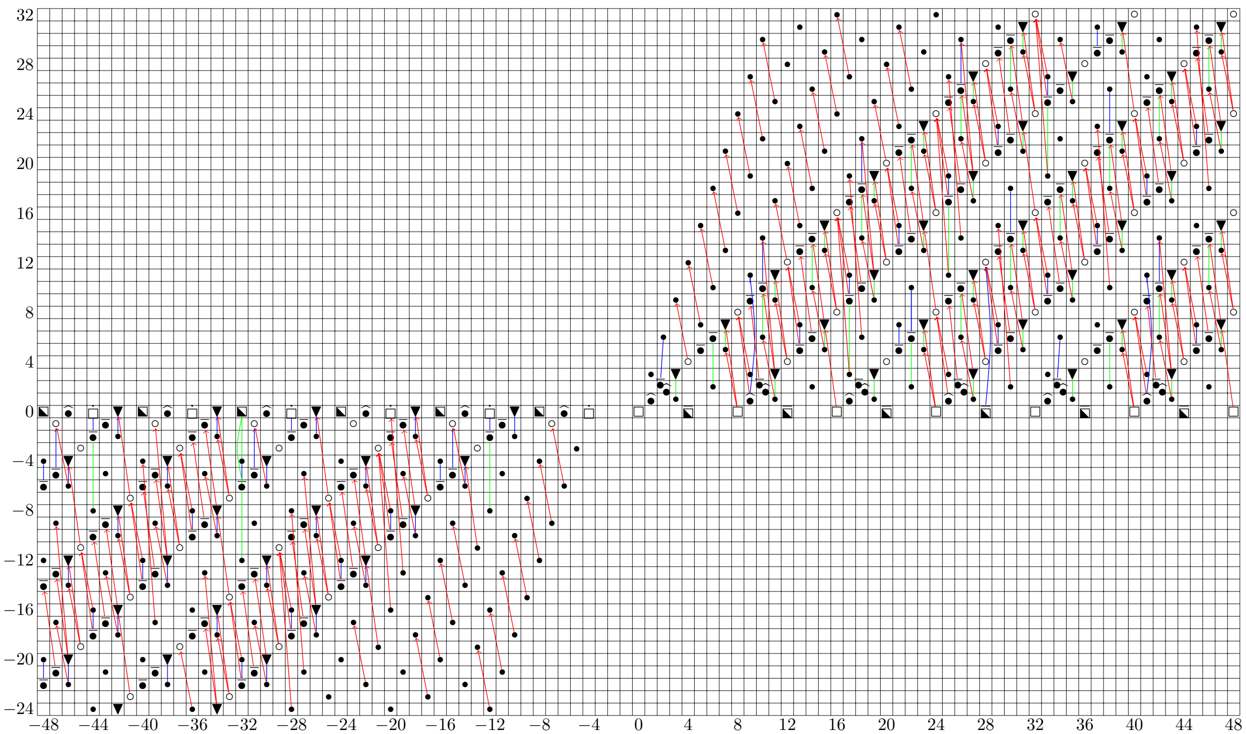


FIGURE 14. The reduced $\underline{E}_4$-term of the slice spectral sequence for the periodic spectrum $K_{\mathbf{H}}$. Differentials are shown in red. Exotic transfers and restrictions are shown in blue and green respectively.

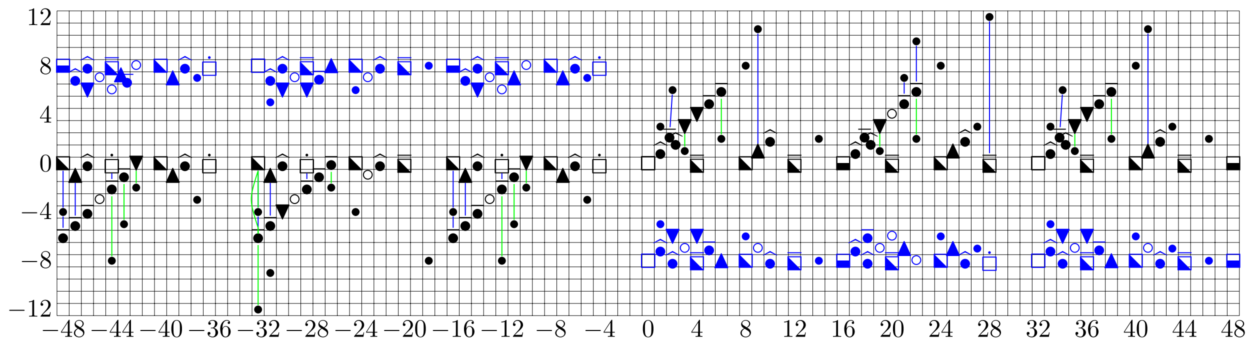


FIGURE 15. The reduced $\underline{E}_\infty$-term of the slice spectral sequence for $K_{\mathbf{H}}$. The exotic Mackey functor extensions lead to the Mackey functors shown in blue in the second and fourth quadrants.

Corollary 41. The $\underline{E}_\infty$-term of the slice spectral sequence for $K_{\mathbf{H}}$. *The surviving elements in the spectral sequence for $K_{\mathbf{H}}$ are shown in Figure 15.*

References

[HHR] Michael A. Hill, Michael J. Hopkins, and Douglas C. Ravenel. The non-existence of elements of Kervaire invariant one. Online at http://arxiv.org/abs/arXiv:0908.3724v2 and on third author's home page.

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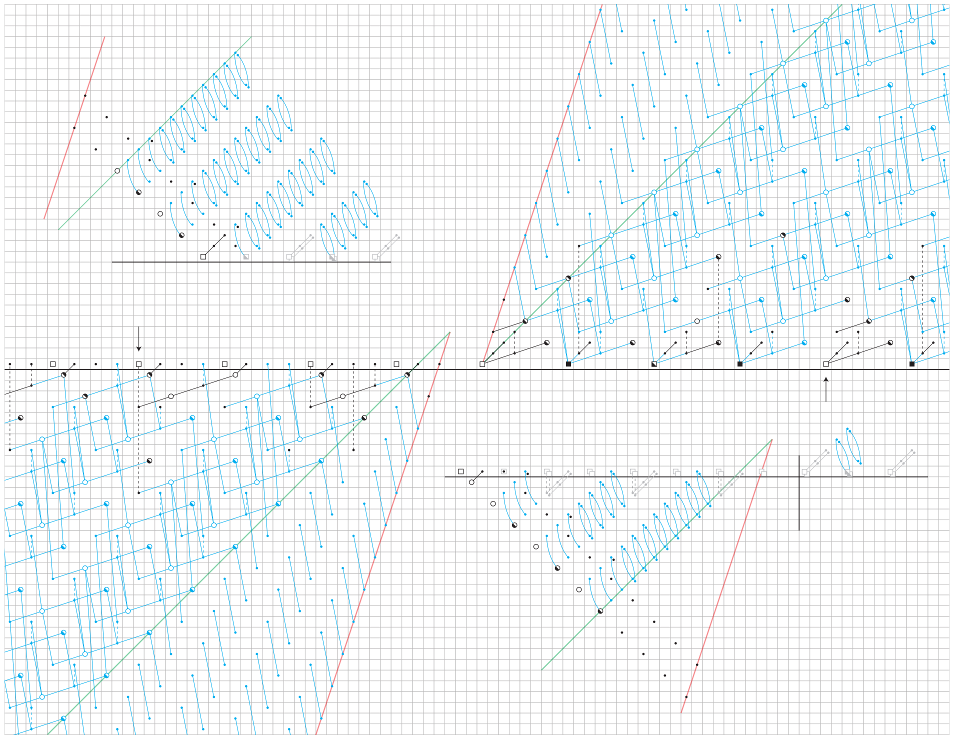

FIGURE 16. The 2008 poster. The first and third quadrants show $\underline{E}_4(G/G)$ with the elements of Prop. 37 excluded. The second quadrant indicates d_3s as in Figures 5 and 6. The fourth quadrant indicates comparable d_3s in the third quadrant of the slice spectral sequence.