

NEW INFINITE FAMILIES IN THE STABLE HOMOTOPY GROUPS OF SPHERES

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ABSTRACT. We identify seven new 192-periodic infinite families of elements in the 2-primary stable homotopy groups of spheres, whose images are nontrivial in the $K(2)$ - as well as the $T(2)$ -local stable stems. We also obtain new information about 2-torsion and 2-divisibility of some of the known 192-periodic infinite families in the stable stems.

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1. INTRODUCTION

Computing the stable homotopy groups of spheres, or stable stems, is one of the central problems in homotopy theory, with many applications in topology, geometry, and algebra. There are two main approaches: low-dimensional computations, which attempt to give a complete description of the stable stems up to a finite range using Adams or Adams-Novikov spectral sequences as the primary tools [MT67, Rav86, KM95, Isa19, IWX23], and chromatic computations, which attempt to pick out large-scale periodic patterns instead [Ada66, Smi77, MRW77, Rav86].

The first large-scale phenomena observed in the stable stems, proven by Serre [Ser53], is that all stable stems above dimension zero are finite abelian groups. This motivated the study of the stable stems one prime at a time.

The next set of important developments were due to Toda [Tod62] and Adams [Ada66]. Their work deduced the existence of $(2p - 2)$ -periodic families of p -torsion elements for primes $p > 2$ and 8-periodic families when $p = 2$ within the stable stems. A decade later, Smith [Smi77] constructed $(2p^2 - 2)$ -periodic families at $p > 3$, and Miller–Ravenel–Wilson [MRW77] constructed $(2p^3 - 2)$ -periodic families at $p > 5$. These examples illustrate an unwritten rule in this subject: *the smaller the prime number p , the harder it is to find exact patterns of p -torsion elements in the stable stems.*

At the prime 2, the chromatic layer 1 patterns (see [Ada66, Rav86]) are more subtle than those at odd primes, and it is evident from the recent results [Bea15, BO16, BG18, BHHM20, BMQ23, BBG⁺23] that the chromatic layer 2 patterns are particularly complicated at $p = 2$.

The 2-local connective spectrum of topological modular forms, tmf , is a formidable tool to explore chromatic height 2 at the prime 2. This is because tmf carries intricate patterns [Bau08, DFHH14, BR21] in its homotopy groups reflecting the patterns in the second chromatic layer of the 2-local stable stems, but is more computationally accessible.

Over the last decade, new techniques have been developed to study the tmf -based Adams spectral sequence [BOSS19, BBT21, BBC23], leading to important and interesting results at chromatic height 2 [BHHM20, Bob20, BE20, BBB⁺21, BMQ23]. In fact, the recent work [BMQ23] completely identifies the image of the Hurewicz homomorphism

$$h_{\mathrm{tmf}} : \pi_* \mathbb{S} \longrightarrow \mathrm{tmf}_*$$

from the stable stems to the coefficients of tmf , thereby proving the existence of new 192-periodic infinite families in the chromatic layer 2 of the 2-local stable stems. In this paper, we show that:

Theorem 1. *For each $m \in \{23, 47, 71, 74, 95, 119, 167\}$ and $k \in \mathbb{N}$, there exists an element of order 2 in dimension $m + 192k$ of the stable stems whose image is trivial under the tmf -Hurewicz homomorphism.*

Remark 1.1. A comparison of our work with known calculations [Isa19, IWX23] suggests that the elements with May names $h_1^3 g$, $h_1^2 \cdot (\Delta h_1 g)$, $h_1^2 \cdot (\Delta^2 h_1 g)$, $d_0 g^3$, $(\Delta h_1)^3 g$, $\Delta^4 h_1^3 g$, and $\Delta^6 h_1^3 g$ in the classical Adams spectral sequence detect the elements in dimension 23, 47, 71, 74, 95, 119, and 167 of Theorem 1, respectively.

Remark 1.2. Let $\eta \in \pi_1 \mathbb{S}$ denote the first Hopf map and let ko denote the connective real K-theory. Then η^3 is a part of an 8-periodic infinite family in chromatic layer 1 which is not detected in the Hurewicz image of ko . From this perspective, the 192-periodic families in Theorem 1 can be regarded as height 2 analogs of the η^3 family.

The spectrum $\mathrm{TMF} \simeq (\Delta^8)^{-1} \mathrm{tmf}$, obtained from tmf by inverting the periodicity generator Δ in degree 192, is a $K(2)$ -local spectrum in the sense of Bousfield [Bou79], where $K(2)$ is the second Morava K-theory. In chromatic homotopy theory, there are also telescopic localizations which are closely related to Bousfield localizations with respect to Morava K-theories. The recent disproof of the telescope conjecture [BHLs23] implies that the natural map from the height 2 telescopic localization to the $K(2)$ -localization of the sphere spectrum

$$\iota : \mathbb{S}_{T(2)} \longrightarrow \mathbb{S}_{K(2)}$$

is not an equivalence. But the chromatic height 2 elements in the Hurewicz image of \mathbf{tmf} do not see this difference as they lift to both the $T(2)$ -local and $K(2)$ -local stable stems. This is because the unit map of \mathbf{TMF}

$$(1) \quad \iota_{\mathbf{tmf}} : \mathbb{S} \longrightarrow \mathbb{S}_{T(2)} \xrightarrow{\iota} \mathbb{S}_{K(2)} \longrightarrow \mathbf{TMF}$$

factors through ι . This argument does not apply to elements listed in [Theorem 1](#) because they are not in the Hurewicz image of \mathbf{tmf}_* . However, we can still show that:

Theorem 2 ([Theorem 3.6](#) and [Theorem 3.14](#)). *All elements listed in [Theorem 1](#) have nonzero images in the $K(2)$ -local and $T(2)$ -local stable stems.*

Although our new infinite families do not contradict the telescope conjecture, they still have significant geometric implications. The groundbreaking work of Kervaire and Milnor [[KM63](#)] directly relates the stable stems to the classification of smooth structures on homotopy spheres. In odd dimensions, the work of Kervaire and Milnor [[KM63](#)], Browder [[Bro69](#)], Hill, Hopkins, and Ravenel [[HHR16](#)], and Wang and Xu [[WX17](#)] implies that exotic spheres exist in every odd dimension except for 1, 3, 5, and 61. In even dimensions, Adams and Toda’s results above imply that exotic spheres exist in at least one quarter of the even dimensions, while the results in [[BHHM20](#), [BMQ23](#)] imply that exotic spheres exist in over half of the even dimensions. Wang and Xu [[WX17](#)] have conjectured that exotic spheres exist in all dimensions except for a small number of low-dimensional exceptions.

[Theorem 1](#) also has implications for exotic spheres. Following Schultz [[Sch85](#)], an exotic sphere is called *very exotic* if it does not bound a parallelizable manifold. Very exotic spheres are more mysterious than exotic spheres which bound parallelizable manifolds; for instance, the latter are always known to admit Riemannian metrics of positive Ricci curvature [[Wra97](#)], while only one very exotic sphere is known to admit such a metric.

In even dimensions, every exotic sphere is a very exotic sphere, but most of the known odd-dimensional exotic spheres are not “very exotic.” The results of [[BHHM20](#), [BMQ23](#)] imply that very exotic spheres exist in at least 37 of the 96 odd congruence classes of dimensions modulo 192. The 6 odd dimensions in [Theorem 1](#) are not covered by those results, so we obtain:

Corollary. Very exotic spheres exist in at least 43 of the 96 odd congruence classes of dimensions modulo 192.

1.1. Methodology. We consider a type 2 spectrum A_1 which is constructed using three cofiber sequences

$$(2) \quad \mathbb{S} \xrightarrow{2} \mathbb{S} \longrightarrow M \xrightarrow{p_1} \Sigma\mathbb{S},$$

$$(3) \quad \Sigma M \xrightarrow{\eta} M \longrightarrow Y \xrightarrow{p_2} \Sigma^2 M,$$

$$(4) \quad \Sigma^2 Y \xrightarrow{v} Y \longrightarrow A_1 \xrightarrow{p_3} \Sigma^3 Y,$$

where v is a choice of a v_1^1 -self-map of Y . The recent work of Viet-Cuong Pham [Pha23], which shows that the tmf -Hurewicz homomorphism

$$(5) \quad h_{\mathrm{tmf}} : \pi_* A_1 \longrightarrow \mathrm{tmf}_* A_1$$

is a surjection, is the starting point of our calculations. We then study long exact sequences associated to the cofiber sequences (2), (3) and (4) using our knowledge of tmf_* [Bau08, DFHH14, BR21], as well as $\mathrm{tmf}_* M$, $\mathrm{tmf}_* Y$, and $\mathrm{tmf}_* A_1$ [BBPX22, Pha23].

By combining this study with our complete knowledge of the Hurewicz image in tmf_* [BMQ23], we identify seven new infinite families of elements in $\pi_* \mathbb{S}$ (listed in Theorem 1) which are in the image of the pinch map

$$(6) \quad p : A_1 \xrightarrow{p_3} \Sigma^3 Y \xrightarrow{p_2} \Sigma^5 M \xrightarrow{p_1} \Sigma^6 \mathbb{S}$$

in stable homotopy. Combining results and techniques of Pham [Pha23], Laures [Lau04], and [BMQ23] shows that these infinite families have nontrivial images in the $K(2)$ -local stable stems. We then use a v_2^{32} -self-map of A_1 [BEM17] to show that these infinite families have nontrivial images in the $T(2)$ -local stable stems, completing the proof of Theorem 2.

The 192-periodic elements in the stable stems constructed in [BMQ23] were all shown to have order at most 8. The tmf -homology calculations of Section 2 lead to new information about the 2-torsion and 2-divisibility of some of the 192-periodic infinite families identified in [BMQ23]. We deduce this from Table 1 using the fact that the elements in the image of p_1 are simple 2-torsion and elements with nontrivial image under i_1 are not 2-divisible, where p_1 and i_1 are the maps defined in (9).

Theorem 3. *An element in the stable stems is simple 2-torsion if it maps to $\Delta^{8k} x$, $k \geq 0$, where*

$$x \in \{\kappa\nu, \bar{\kappa}^2\eta^2, \eta\Delta\bar{\kappa}^2, 4\Delta^2\bar{\kappa}, \bar{\kappa}^4, \eta^2\Delta^2\bar{\kappa}^2, 2\Delta^4 \cdot 2\bar{\kappa}, 4\Delta^6\bar{\kappa}\},$$

and not 2-divisible if it maps to $\Delta^{8k} x$, $k \geq 0$, where

$$x \in \{\nu^2, \nu^3, \bar{\kappa}\nu, q\eta, \bar{\kappa}^2\eta^2, \eta^2\Delta^2\nu, \nu\Delta^2\nu, \nu\Delta^2\nu^2, \nu\Delta^2\nu^3, 4\Delta^2\bar{\kappa}, \bar{\kappa}^4, \eta\Delta\bar{\kappa}^3, \eta^2\Delta^2\bar{\kappa}^2, \bar{\kappa}^5, \nu^3\Delta^4, \eta\Delta\bar{\kappa}^4, 2\Delta^4\bar{\kappa}, \eta\Delta\bar{\kappa}^5, \eta^2\Delta^2\kappa, \nu\Delta^6\nu^2, \nu\Delta^6\eta\epsilon\}.$$

Organization of the paper. In Section 2, we perform the tmf -homology calculations necessary in Section 3 to prove Theorem 1 and Theorem 2.

For the purpose of this paper, a reader may find [DFHH14, Part I, Ch. 12] convenient for looking up the homotopy groups of tmf , where the generators in the Hurewicz image are marked with colored dots. We refer to [BBPX22] for explicit descriptions of $\mathrm{tmf}_* M$, $\mathrm{tmf}_* Y$, and $\mathrm{tmf}_* A_1$.

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2. tmf-HOMOLOGY CALCULATIONS

Using our knowledge of tmf_* [[Bau08](#), [BR21](#)], tmf_*M , tmf_*Y , and tmf_*A_1 [[BBPX22](#)], we will compute the maps i_k and p_k in the long exact sequences

$$(7) \quad \cdots \longrightarrow \mathrm{tmf}_k Y \xrightarrow{i_3} \mathrm{tmf}_k A_1 \xrightarrow{p_3} \mathrm{tmf}_{k-3} Y \xrightarrow{v_*} \cdots$$

$$(8) \quad \cdots \longrightarrow \mathrm{tmf}_{k-3} M \xrightarrow{i_2} \mathrm{tmf}_{k-3} Y \xrightarrow{p_2} \mathrm{tmf}_{k-5} M \xrightarrow{\eta_*} \cdots$$

$$(9) \quad \cdots \longrightarrow \mathrm{tmf}_{k-5} \xrightarrow{i_1} \mathrm{tmf}_{k-5} M \xrightarrow{p_1} \mathrm{tmf}_{k-6} \xrightarrow{2} \cdots$$

associated to the cofiber sequences (2), (3) and (4), respectively. This is the technical core of the paper and requires careful bookkeeping using Adams–Novikov spectral sequences. In our arguments, we ignore v_1 -periodic classes for reasons we will now explain.

2.1. Suppression of v_1 -periodic families.

Note that the element $c_4 \in \mathrm{tmf}_*$ is the v_1 -periodicity generator as it maps to $v_1^4 \in k(1)_*$ under the composition

$$\mathrm{tmf} \longrightarrow \mathrm{ko} \longrightarrow k(1),$$

where $k(1)$ is the connective height 1 Morava K-theory (see [[DM10](#), [BR21](#)]). It is well-known that

$$v_1^{-1}\mathrm{tmf} := \mathrm{colim} \{ \mathrm{tmf} \xrightarrow{c_4} \mathrm{tmf} \xrightarrow{c_4} \cdots \} \simeq \mathrm{KO}[j^{-1}],$$

where $j = \Delta/c_4^3$ (see [[Lau04](#), Corollary 3]).

Definition 2.1. For any spectrum X , define the v_1 -torsion part of $\mathrm{tmf}_*(X)$ as the kernel

$$\mathrm{tmf}_*(X)^{\mathrm{tor}} := \ker \left(\ell : \mathrm{tmf}_* X \longrightarrow v_1^{-1}\mathrm{tmf}_* X \right)$$

of the v_1 -localization map.

Lemma 2.2. For any nonzero element $a \in \mathrm{tmf}_* A_1$ we have

- (a) $p_3(a) \in \mathrm{tmf}_*(Y)^{\mathrm{tor}}$,
- (b) $p_2(p_3(a)) \in \mathrm{tmf}_*(M)^{\mathrm{tor}}$, and
- (c) $p_1(p_2(p_3(a))) \in \mathrm{tmf}_*(\mathbb{S})^{\mathrm{tor}}$.

Proof. Since A_1 is a type 2 spectrum, it follows that $v_1^{-1}\mathbf{tmf}_*A_1 = 0$. Therefore, from the commutative diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & \mathbf{tmf}_k Y & \xrightarrow{i_3} & \mathbf{tmf}_k A_1 & \xrightarrow{p_3} & \mathbf{tmf}_{k-3} Y & \longrightarrow & \dots \\ & & \ell \downarrow & & \downarrow \ell & & \downarrow \ell & & \\ \dots & \longrightarrow & v_1^{-1}\mathbf{tmf}_k Y & \xrightarrow{i_3} & v_1^{-1}\mathbf{tmf}_k A_1 & \xrightarrow{p_3} & v_1^{-1}\mathbf{tmf}_{k-3} Y & \longrightarrow & \dots \end{array}$$

of long exact sequences, we get

$$\ell(p_3(a)) = p_3(\ell(a)) = p_3(0) = 0$$

which means $p_3(a) \in \mathbf{tmf}_*(Y)^{\text{tor}}$.

For part (b), we consider the diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & \mathbf{tmf}_{k-3} M & \xrightarrow{i_2} & \mathbf{tmf}_{k-3} Y & \xrightarrow{p_2} & \mathbf{tmf}_{k-5} M & \longrightarrow & \dots \\ & & \ell \downarrow & & \downarrow \ell & & \downarrow \ell & & \\ \dots & \longrightarrow & v_1^{-1}\mathbf{tmf}_{k-3} M & \xrightarrow{i_2} & v_1^{-1}\mathbf{tmf}_{k-3} Y & \xrightarrow{p_2} & v_1^{-1}\mathbf{tmf}_{k-5} M & \longrightarrow & \dots \end{array}$$

of long exact sequences, and observe

$$\ell(p_2(p_3(a))) = p_2(\ell(p_3(a))) = p_2(0) = 0$$

which implies $p_1(p_3(a)) \in \mathbf{tmf}_*(M)^{\text{tor}}$.

A similar study of a commutative diagram for the cofiber sequence (2) proves (c). \square

Definition 2.3. For any spectrum X , define the v_1 -periodic part of \mathbf{tmf}_*X as the cokernel

$$\mathbf{tmf}_*(X)^{\text{per}} := \text{coker} \left(\mathbf{tmf}_*(X)^{\text{tor}} \hookrightarrow \mathbf{tmf}_*X \right)$$

of the natural inclusion map.

Remark 2.4 (Exactness of v_1 -periodic part). Direct calculations show that the long exact sequences in \mathbf{tmf} -homology associated to the cofiber sequences (2), (3), and (4) give rise to long exact sequences on v_1 -periodic parts. The authors are unaware if this is a part of a general pattern, i.e., whether $\mathbf{tmf}_*(-)^{\text{per}}$ is a homology theory.

Lemma 2.5. If $p_2(p_3(a)) = 0$ in \mathbf{tmf}_*M , then there exists a class

$$m_0 \in \mathbf{tmf}_*(M)^{\text{tor}}$$

such that $i_2(m_0) = p_3(a)$.

Proof. The map η induces a map

$$\eta_*^{\text{per}} : \mathbf{tmf}_{*-1}(M)^{\text{per}} \longrightarrow \mathbf{tmf}_*(M)^{\text{per}},$$

and we have a commutative diagram

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & \mathbf{tmf}_{*-1}M & \xrightarrow{\eta_*} & \mathbf{tmf}_*M & \xrightarrow{i_2} & \mathbf{tmf}_*Y \longrightarrow \dots \\
 & & \pi_1 \downarrow & & \downarrow \pi_2 & & \downarrow \pi_3 \\
 \dots & \longrightarrow & \mathbf{tmf}_{*-1}M^{\text{per}} & \xrightarrow{\eta_*^{\text{per}}} & \mathbf{tmf}_*(M)^{\text{per}} & \xrightarrow{i_2} & \mathbf{tmf}_*(Y)^{\text{per}} \longrightarrow \dots
 \end{array}$$

in which the vertical maps are surjections.

If $p_2(p_3(a)) = 0$ then there exists $m \in \mathbf{tmf}_*M$ such that $i_2(m) = p_3(a)$. By [Lemma 2.2](#)

$$i_2(\pi_2(m)) = \pi_3(i_2(m)) = \pi_3(p_3(a)) = 0,$$

therefore, by [Remark 2.4](#), $\pi_2(m) = \eta_*^{\text{per}}(m')$ for some $m' \in \mathbf{tmf}_{*-1}$. Let $m'' \in \mathbf{tmf}_{*-1}M$ be a lift of m' along π_1 . It is easy to see that

$$m_0 = m - \eta_*(m'') \in \mathbf{tmf}_*(M)^{\text{tor}}$$

and $i_2(m_0) = i_2(m - \eta_*(m'')) = i_2(m) - i_2(\eta_*(m'')) = i_2(m) = p_3(a)$. \square

A similar argument leads to the following result.

Lemma 2.6. If $p_1(p_2(p_3(a))) = 0$ in \mathbf{tmf}_* then there exists a class

$$s \in \mathbf{tmf}_*^{\text{tor}}$$

such that $i_1(s) = p_2(p_3(a))$.

2.2. From \mathbf{tmf}_*Y to \mathbf{tmf}_*M .

An element $y \in \mathbf{tmf}_{k-3}Y$ is in the image of p_3 for some version of A_1 if and only if

$$v_1 \cdot y = 0 \in \mathbf{tmf}_{k-1}Y$$

for a choice of v_1 . Since the action of all v_1 -self-maps on \mathbf{tmf}_*Y have been identified on each generator [[BBPX22](#), Figs. 22, 23], the image of p_3 is easily determined; we list these elements in the leftmost column of [Table 1](#).

Notation 2.7. Let $s_{i,j}$, $m_{i,j}$ and $y_{i,j}$ denote elements of \mathbf{tmf}_* , \mathbf{tmf}_*M , and \mathbf{tmf}_*Y , respectively, which are detected in filtration $(j, j+i)$ of the Adams-Novikov spectral sequence (11). In the bidegrees that we are interested in, there is only one element which is v_1 -torsion and nonzero, thus $s_{i,j}$, $m_{i,j}$, and $y_{i,j}$ represents unique elements up to higher Adams-Novikov filtration.

Next, we determine the effect of the map p_2 on the classes in $\text{img}(p_3) \subset \mathbf{tmf}_{*-3}Y$. In particular, we are interested in identifying those classes whose images under p_2 are nonzero. We will use the long exact sequence (8)

$$\dots \longrightarrow \mathbf{tmf}_{k-3}M \xrightarrow{i_2} \mathbf{tmf}_{k-3}Y \xrightarrow{p_2} \mathbf{tmf}_{k-5}M \xrightarrow{\eta_*} \dots$$

By [Lemma 2.2](#) and [Lemma 2.5](#), it suffices to study the short exact sequence

$$(10) \quad \mathbf{C}_{k-3}^{\text{tor}} \hookrightarrow \mathbf{tmf}_{k-3}(Y)^{\text{tor}} \xrightarrow{p_2} \mathbf{K}_{k-5}^{\text{tor}},$$

where $C_{k-3} := C_{k-3}^{\text{tor}}(Y)$ is the cokernel of η_* in (8) and $K_{k-5}^{\text{tor}} = K_{k-5}^{\text{tor}}(Y)$ is the kernel of η_* in (8) restricted to v_1 -torsion (we drop Y from notation for convenience). We employ some standard techniques in our analysis which are listed below.

Technique 1 (Vanishing K). If $K_{k-5}^{\text{tor}} = 0$ in (10), then

$$p_2(y) = 0$$

for any $y \in \text{tmf}_{k-3}(Y)^{\text{tor}}$.

Application 1. We employ [Technique 1](#) to conclude that the following elements map to zero under p_2 :

- | | | | |
|--------------|---------------|----------------|----------------|
| • $Y_{3,1}$ | • $Y_{54,2}$ | • $Y_{85,17}$ | • $Y_{117,3}$ |
| • $Y_{6,2}$ | • $Y_{60,10}$ | • $Y_{86,12}$ | • $Y_{117,13}$ |
| • $Y_{14,2}$ | • $Y_{60,12}$ | • $Y_{90,14}$ | • $Y_{123,11}$ |
| • $Y_{18,2}$ | • $Y_{65,7}$ | • $Y_{91,13}$ | • $Y_{132,16}$ |
| • $Y_{21,3}$ | • $Y_{65,13}$ | • $Y_{96,14}$ | • $Y_{137,17}$ |
| • $Y_{29,5}$ | • $Y_{66,2}$ | • $Y_{97,9}$ | • $Y_{142,18}$ |
| • $Y_{34,6}$ | • $Y_{69,3}$ | • $Y_{101,15}$ | • $Y_{143,15}$ |
| • $Y_{39,7}$ | • $Y_{75,13}$ | • $Y_{105,21}$ | • $Y_{148,18}$ |
| • $Y_{40,6}$ | • $Y_{76,10}$ | • $Y_{106,16}$ | • $Y_{161,7}$ |
| • $Y_{45,9}$ | • $Y_{80,16}$ | • $Y_{111,17}$ | • $Y_{165,3}$ |
| • $Y_{51,1}$ | • $Y_{81,11}$ | • $Y_{112,12}$ | • $Y_{168,22}$ |

Technique 2 (Vanishing C). Suppose $y \in \text{tmf}_{k-3}(Y)^{\text{tor}}$ is a nonzero element and $C_{k-3}^{\text{tor}} = 0$, then

$$p_2(y) \neq 0$$

in (10). Further, if $\text{rank}_{\mathbb{F}_2}(K_{k-5}^{\text{tor}}) = 1$ then the image of y is the unique nonzero element of K_{k-5}^{tor} .

Application 2. We employ [Technique 2](#) to determine the following:

- | | |
|------------------------------|-------------------------------|
| • $p_2(y_{8,2}) = m_{6,2}$ | • $p_2(y_{62,2}) = m_{60,12}$ |
| • $p_2(y_{11,3}) = m_{9,3}$ | • $p_2(y_{68,2}) = m_{66,2}$ |
| • $p_2(y_{23,3}) = m_{21,5}$ | • $p_2(y_{74,4}) = m_{72,6}$ |
| • $p_2(y_{26,4}) = m_{24,6}$ | • $p_2(y_{77,5}) = m_{75,13}$ |
| • $p_2(y_{44,8}) = m_{42,8}$ | • $p_2(y_{82,6}) = m_{80,16}$ |
| • $p_2(y_{59,3}) = m_{57,3}$ | • $p_2(y_{83,3}) = m_{81,3}$ |

- $p_2(y_{87,7}) = \mathfrak{m}_{85,13}$
- $p_2(y_{88,6}) = \mathfrak{m}_{86,12}$
- $p_2(y_{92,8}) = \mathfrak{m}_{90,10}$
- $p_2(y_{93,3}) = \mathfrak{m}_{91,9}$
- $p_2(y_{98,4}) = \mathfrak{m}_{96,6}$
- $p_2(y_{108,10}) = \mathfrak{m}_{106,16}$
- $p_2(y_{113,7}) \neq 0$
- $p_2(y_{119,3}) = \mathfrak{m}_{117,3}$
- $p_2(y_{127,15}) = \mathfrak{m}_{125,21}$
- $p_2(y_{133,11}) \neq 0$
- $p_2(y_{155,3}) = \mathfrak{m}_{153,3}$
- $p_2(y_{158,16}) = \mathfrak{m}_{156,18}$
- $p_2(y_{167,3}) = \mathfrak{m}_{165,3}$
- $p_2(y_{170,4}) = \mathfrak{m}_{168,6}$

Technique 3 (Action of \mathbf{tmf}_*). The maps i_2 and p_2 in (8) and (10) are \mathbf{tmf}_* -linear, i.e.,

- (1) $p_2(t \cdot y) = t \cdot p_2(y)$,
- (2) $i_2(t \cdot m) = t \cdot i_2(m)$

for all $t \in \mathbf{tmf}_*$, $m \in \mathbf{tmf}_*M$ and $y \in \mathbf{tmf}_*Y$.

Application 3. We use [Technique 3](#) to show that

- $p_2(y_{102,10}) = p_2(\bar{\kappa} \cdot y_{82,6}) = \bar{\kappa} \cdot p_2(y_{82,6}) = \bar{\kappa} \cdot \mathfrak{m}_{80,16} = \mathfrak{m}_{100,20}$ which forces $p_2(y_{102,2}) = 0$,
- $p_2(y_{118,8}) = p_2(\bar{\kappa} \cdot y_{98,4}) = \bar{\kappa} \cdot p_2(y_{98,4}) = \bar{\kappa} \cdot \mathfrak{m}_{96,6} = \mathfrak{m}_{116,10}$,
- $p_2(y_{138,12}) = p_2(\bar{\kappa} \cdot y_{118,8}) = \bar{\kappa} \cdot p_2(y_{118,8}) = \bar{\kappa} \cdot \mathfrak{m}_{116,10} = \mathfrak{m}_{136,14}$,
- $p_2(y_{153,15}) = p_2(\bar{\kappa} \cdot y_{133,11}) = \bar{\kappa} \cdot p_2(y_{133,11}) = \bar{\kappa} \cdot \mathfrak{m}_{131,17} = \mathfrak{m}_{151,21}$ which forces $p_2(y_{153,11}) = 0$.

The next few techniques use the fact that the \mathbf{tmf} -homology of Y and M are calculated in [\[BBPX22\]](#) using the Adams–Novikov spectral sequence

$$(11) \quad (-)_{\mathbf{E}_2}^{s,t} := \text{Ext}_{\Gamma}^{s,t}(A, \pi_*(\mathbf{tmf} \wedge X(4) \wedge (-))) \implies \mathbf{tmf}_{t-s}(-),$$

where the spectrum $X(4)$ and the Hopf algebroid (A, Γ) are as described in [\[BBPX22, §2.1\]](#).

Technique 4 (Analysis of \mathbf{E}_2 -pages). Corresponding to the cofiber sequence (3), there is a long exact sequence

$$(12) \quad \dots \longrightarrow M_{\mathbf{E}_2}^{s,t} \xrightarrow{\hat{i}_2} Y_{\mathbf{E}_2}^{s,t} \xrightarrow{\hat{p}_2} M_{\mathbf{E}_2}^{s,t-2} \xrightarrow{\hat{\eta}} \dots$$

of \mathbf{E}_2 -pages of Adams–Novikov spectral sequences. Suppose $y \in \mathbf{tmf}_{k-3}Y$ is detected by $\hat{y} \in Y_{\mathbf{E}_2}^{s,k-3+s}$.

- (1) If $m \in \mathbf{tmf}_{k-3}M$ is detected by $\hat{m} \in M_{\mathbf{E}_2}^{s,k-3+s}$ such that
 - (a) $\hat{i}_2(\hat{m}) = \hat{y}$ and

(b) \widehat{m} is a permanent cycle,

then $i_2(m) = y$.

(2) If $m \in \mathbf{tmf}_{k-5}\mathbf{M}$ is detected by $\widehat{m} \in {}^M\mathbf{E}_2^{s,k-5+s}$ such that

(a) $\widehat{p}_2(y) = \widehat{m}$ and

(b) \widehat{m} is a permanent cycle,

then $p_2(y) = m$.

Application 4. We use [Technique 4](#) to determine

- $i_2(\mathbf{m}_{20,4}) = y_{20,4}$ which forces $p_2(y_{20,2}) = \mathbf{m}_{18,2}$,
- $i_2(\mathbf{m}_{103,1}) = y_{103,1}$ which forces $p_2(y_{103,7}) = \mathbf{m}_{101,2}$,
- $i_2(\mathbf{m}_{150,2}) = y_{150,2}$.

Definition 2.8. We say an element $x \in \mathbf{tmf}_*(X)$ has Adams-Novikov filtration s , denoted $\mathbf{AF}(x) = s$, if it is detected by an element

$$\widehat{x} \in {}^X\mathbf{E}_2^{s,*+s}$$

in the \mathbf{E}_2 -page of (11).

Remark 2.9. In [Notation 2.7](#), the Adams filtration of elements $\mathbf{s}_{i,j}$, $\mathbf{m}_{i,j}$ and $\mathbf{y}_{i,j}$ equals j .

Our next technique follows from the fact that maps of spectra cannot decrease Adams–Novikov filtration.

Technique 5 (Adams–Novikov filtration argument). Suppose $y \in \mathbf{tmf}_{k-3}\mathbf{Y}$ is a nonzero element and $m \in \mathbf{tmf}_*\mathbf{M}$.

- (1) If $\mathbf{AF}(y) > \mathbf{AF}(m)$, then $p_2(y) \neq m$.
- (2) If $\mathbf{AF}(y) < \mathbf{AF}(m)$, then $i_2(m) \neq y$.

Application 5. We use [Technique 5](#) to conclude that

- $i_2(\mathbf{m}_{35,5}) \neq y_{35,3}$ and $p_2(y_{35,3}) \neq \mathbf{m}_{33,1}$ which forces $p_2(y_{35,3}) = \mathbf{m}_{33,3}$,
- $i_2(\mathbf{m}_{45,5}) \neq y_{45,3}$ which forces $p_2(y_{45,3}) = \mathbf{m}_{43,9}$ and $p_2(y_{45,9}) = 0$,
- $i_2(\mathbf{m}_{55,9}) \neq y_{55,7}$ which forces $p_2(y_{55,7}) = \mathbf{m}_{53,7}$,
- $i_2(y_{56,6}) \neq \mathbf{m}_{54,2}$ which forces $p_2(y_{56,2}) = \mathbf{m}_{54,2}$,
- $p_2(y_{57,11}) \neq \mathbf{m}_{55,9}$ which forces $p_2(y_{57,11}) = 0$,
- $p_2(y_{71,9}) \neq \mathbf{m}_{69,3}$ which forces $p_2(y_{71,9}) = 0$ and $p_2(y_{71,3}) = \mathbf{m}_{69,3}$,
- $p_2(y_{107,11}) \neq \mathbf{m}_{105,3}$ which forces $p_2(y_{107,3}) = \mathbf{m}_{105,3}$ and $p_2(y_{107,11}) = \mathbf{m}_{105,17}$,

- $p_2(y_{113,7}) \neq m_{111,3}$ which forces $p_2(y_{113,7}) = m_{111,13}$,
- $p_2(y_{122,14}) \neq m_{120,6}$, which forces $p_2(y_{122,14}) = 0$ and $p_2(y_{122,4}) = m_{120,6}$,
- $p_2(y_{133,11}) \neq m_{133,7}$ which forces $p_2(y_{133,11}) = m_{131,17}$.

Technique 6 (Geometric boundary theorem [Beh12, Lemma A.4.1 (5)]). Consider the maps of the Adams-Novikov spectral sequences induced by (3)

$$\Sigma M E_r^{s,*+s} \xrightarrow{\hat{\eta}} M E_r^{s,*+s} \xrightarrow{\hat{i}_2} Y E_r^{s,*+s} \xrightarrow{\hat{p}_2} \Sigma^2 M E_r^{s,*+s}.$$

Suppose $\hat{m} \in M E_r^{s,*+s}$ such that

- $d_r(\hat{m}) = \hat{\eta}(\hat{m}')$,
- $\hat{i}_2(\hat{m}) = \hat{y}$ is a nonzero permanent cycle,

then $\hat{p}_2(\hat{y}) = \hat{m}'$.

Application 6. We use [Technique 6](#) in the following arguments:

- Since $d_5(m_{50,6}) = \hat{\eta}(m_{48,6})$ and $\hat{i}_2(m_{50,6}) = y_{50,6}$ is a nonzero permanent cycle, we get $p_2(y_{50,6}) = m_{48,6}$. Consequently, $p_2(y_{50,4}) = 0$.
- Since $d_5(m_{70,10}) = \hat{\eta}(m_{68,10})$ and $\hat{i}_2(m_{70,10}) = y_{70,10}$ is a permanent cycle, we get $p_2(y_{70,10}) = m_{48,10}$. Consequently, $p_2(y_{70,8}) = 0$. Alternatively, this case follows from the previous case using $\bar{\kappa}$ -linearity.
- Since $d_5(m_{128,14}) = \hat{\eta}(m_{126,20})$ and $\hat{i}_2(m_{128,14}) = y_{128,14}$ is a permanent cycle, we get $p_2(y_{128,14}) = m_{126,20}$. Alternatively, this follows using $\bar{\kappa}$ -linearity from the fact that $p_2(y_{108,10}) = m_{106,16}$ which was established earlier using [Technique 2](#).

2.3. From tmf_*M to tmf_* .

All the techniques above have analogs corresponding to the cofiber sequence (2). We use them to study the short exact sequence

$$(13) \quad C_{k-3}^{\mathrm{tor}} \hookrightarrow \mathrm{tmf}_{k-3}(M)^{\mathrm{tor}} \twoheadrightarrow K_{k-4}^{\mathrm{tor}},$$

where C_{k-3}^{tor} is the cokernel of i_1 and K_{k-4}^{tor} is the kernel of p_1 , both restricted to v_1 -torsion.

Example 2.10. We use the analog of [Technique 1](#) to determine

- | | |
|------------------------------|--------------------------------|
| • $i_1(s_{6,2}) = m_{6,2}$ | • $i_1(s_{48,0}) = m_{48,6}$ |
| • $i_1(s_{9,3}) = m_{9,3}$ | • $i_1(s_{53,7}) = m_{53,7}$ |
| • $i_1(s_{21,5}) = m_{21,5}$ | • $i_1(s_{57,3}) = m_{57,3}$ |
| • $i_1(s_{24,0}) = m_{24,0}$ | • $i_1(s_{60,12}) = m_{60,12}$ |

- $i_1(\mathfrak{s}_{68,4}) = \mathfrak{m}_{68,10}$
- $i_1(\mathfrak{s}_{72,0}) = \mathfrak{m}_{72,6}$
- $i_1(\mathfrak{s}_{75,3}) = \mathfrak{m}_{75,13}$
- $i_1(\mathfrak{s}_{80,16}) = \mathfrak{m}_{80,16}$
- $i_1(\mathfrak{s}_{85,13}) = \mathfrak{m}_{85,13}$
- $i_1(\mathfrak{s}_{90,10}) = \mathfrak{m}_{90,10}$
- $i_1(\mathfrak{s}_{96,0}) = \mathfrak{m}_{96,6}$
- $i_1(\mathfrak{s}_{100,20}) = \mathfrak{m}_{100,20}$
- $p_1(\mathfrak{m}_{105,3}) = 0$
- $p_1(\mathfrak{m}_{105,17}) = 0$
- $i_1(\mathfrak{s}_{116,4}) = \mathfrak{m}_{116,10}$
- $i_1(\mathfrak{s}_{120,0}) = \mathfrak{m}_{120,6}$
- $i_1(\mathfrak{s}_{153,3}) = \mathfrak{m}_{153,3}$
- $i_1(\mathfrak{s}_{168,0}) = \mathfrak{m}_{168,6}$.

Example 2.11. We use the analog of [Technique 2](#) to determine

- $p_1(\mathfrak{m}_{18,2}) = \mathfrak{s}_{17,2}$
- $p_1(\mathfrak{m}_{43,9}) = \mathfrak{s}_{42,11}$
- $p_1(\mathfrak{m}_{69,3}) = \mathfrak{s}_{68,4}$
- $p_1(\mathfrak{m}_{81,3}) = \mathfrak{s}_{80,16}$
- $p_1(\mathfrak{m}_{86,12}) = \mathfrak{s}_{85,13}$
- $p_1(\mathfrak{m}_{91,9}) = \mathfrak{s}_{90,10}$
- $p_1(\mathfrak{m}_{101,7}) = \mathfrak{s}_{100,20}$
- $p_1(\mathfrak{m}_{106,16}) = \mathfrak{s}_{105,17}$
- $p_1(\mathfrak{m}_{126,20}) = \mathfrak{m}_{125,21}$
- $p_1(\mathfrak{m}_{131,17}) = \mathfrak{s}_{130,18}$
- $p_1(\mathfrak{m}_{151,21}) = \mathfrak{s}_{150,22}$
- $p_1(\mathfrak{m}_{165,3}) = \mathfrak{s}_{164,4}$.

Example 2.12. We use the analog of [Technique 3](#) to deduce that

- $p_2(\mathfrak{m}_{111,13}) = p_2(\bar{\kappa} \cdot \mathfrak{m}_{91,9}) = \bar{\kappa} \cdot \mathfrak{s}_{90,10} = \mathfrak{s}_{110,14}$,
- $i_1(\mathfrak{s}_{136,8}) = i_1(\bar{\kappa} \cdot \mathfrak{s}_{116,4}) = \bar{\kappa} \cdot \mathfrak{m}_{116,10} = \mathfrak{m}_{136,14}$,
- $i_1(\mathfrak{s}_{156,12}) = i_1(\bar{\kappa} \cdot \mathfrak{s}_{136,8}) = \bar{\kappa} \cdot \mathfrak{m}_{136,14} = \mathfrak{m}_{156,18}$.

Example 2.13. We use the analog of [Technique 4](#) to argue that:

- $i_2(\mathfrak{s}_{54,2}) = \mathfrak{m}_{54,2}$.

Example 2.14. The analog of [Technique 5](#) is used to deduce that

- $i_1(\mathfrak{s}_{9,3}) \neq \mathfrak{m}_{9,1}$ which forces $i_1(\mathfrak{s}_{9,3}) = \mathfrak{m}_{9,3}$,
- $p_1(\mathfrak{m}_{33,3}) \neq \mathfrak{s}_{32,2}$ which forces $i_1(\mathfrak{s}_{33,3}) = \mathfrak{m}_{33,3}$,
- $i_1(\mathfrak{s}_{42,10}) \neq \mathfrak{m}_{42,8}$ which forces $i_1(\mathfrak{s}_{42,10}) = \mathfrak{m}_{42,10}$,
- $i_1(\mathfrak{s}_{60,12}) \neq \mathfrak{m}_{60,7}$ which forces $p_1(\mathfrak{m}_{60,7}) = \mathfrak{s}_{59,7}$,
- $p_1(\mathfrak{m}_{66,8}) \neq \mathfrak{s}_{65,3}$ and $i_1(\mathfrak{s}_{66,10}) \neq \mathfrak{m}_{66,8}$ which forces $p_1(\mathfrak{m}_{66,8}) = \mathfrak{s}_{65,9}$,
which along with $i_1(\mathfrak{s}_{66,10}) \neq \mathfrak{m}_{66,2}$ forces $p_1(\mathfrak{m}_{66,2}) = \mathfrak{s}_{65,3}$,
- $i_1(\mathfrak{s}_{105,11}) \neq \mathfrak{m}_{105,3}$ which forces $i_1(\mathfrak{s}_{105,11}) = \mathfrak{m}_{105,11}$ and consequently
 $i_1(\mathfrak{s}_{105,3}) = \mathfrak{m}_{105,3}$,
- $i_1(\mathfrak{s}_{117,5}) \neq \mathfrak{m}_{117,3}$ which forces $p_1(\mathfrak{m}_{117,3}) = \mathfrak{s}_{116,4}$,
- $p_2(\mathfrak{m}_{125,21}) \neq \mathfrak{s}_{124,6}$ which forces $i_1(\mathfrak{s}_{125,21}) = \mathfrak{m}_{125,21}$.

2.4. Summary Table. We summarize our calculations in [Table 1](#) as follows. The leftmost column lists the image of p_3 in \mathbf{tmf}_*Y . We determine their image in column 2 and indicate the technique used, among [Technique 1](#) through [Technique 6](#), in column 3.

We calculate the image under p_1 of nonzero elements in column 2 and record them in column 4. If the image is zero, we identify a v_1 -torsion element which is its lift along i_1 and record it in column 5. We indicate the technique in column 6.

Note that the elements listed in columns 4 and 5 are elements of \mathbf{tmf}_* . We record their familiar names from [\[DFHH14\]](#) in column 7.

Table 1: Detecting elements in \mathbf{tmf}_*

$\mathbf{img}(p_3)$	$\mathbf{img}(p_2)$	(T)	$\mathbf{img}(p_1)$	$i_1^{-1}(-)$	(T)	name in \mathbf{tmf}_*
Y _{3,1}	0	(1)				
Y _{6,2}	0	(1)				
Y _{8,2}	$\mathbf{m}_{6,2}$	(2)	0	$\mathbf{s}_{6,2}$	(1)	ν^2
Y _{11,3}	$\mathbf{m}_{9,3}$	(2)	0	$\mathbf{s}_{9,3}$	(5)	ν^3
Y _{14,2}	0	(1)				
Y _{18,2}	0	(1)				
Y _{20,2}	$\mathbf{m}_{18,2}$	(4)	$\mathbf{s}_{17,2}$		(2)	$\kappa\nu$
Y _{21,3}	0	(1)				
Y _{23,3}	$\mathbf{m}_{21,5}$	(2)	0	$\mathbf{s}_{21,5}$	(1)	$\bar{\kappa}\nu$
Y _{26,4}	$\mathbf{m}_{24,6}$	(2)	0	$\mathbf{s}_{24,0}$	(1)	8Δ
Y _{29,5}	0	(1)				
Y _{34,6}	0	(1)				
Y _{35,3}	$\mathbf{m}_{33,3}$	(5)	0	$\mathbf{s}_{33,3}$	(5)	$q\eta$
Y _{39,7}	0	(1)				
Y _{40,6}	0	(1)				
Y _{44,8}	$\mathbf{m}_{42,10}$	(2)	0	$\mathbf{s}_{42,10}$	(5)	$\bar{\kappa}^2\eta^2$
Y _{45,3}	$\mathbf{m}_{43,9}$	(5)	$\mathbf{s}_{42,10}$		(2)	$\bar{\kappa}^2\eta^2$
Y _{45,9}	0	(1)				
Y _{50,4}	0	(6)				
Y _{50,6}	$\mathbf{m}_{48,6}$	(6)	0	$\mathbf{s}_{48,0}$	(1)	$4\Delta^2$
Y _{51,1}	0	(1)				
Y _{54,2}	0	(1)				
Y _{55,7}	$\mathbf{m}_{53,7}$	(5)	0	$\mathbf{s}_{53,7}$	(1)	$\eta^2\Delta^2\nu$
Y _{56,2}	$\mathbf{m}_{54,2}$	(5)	0	$\mathbf{s}_{54,2}$	(4)	$\nu\Delta^2\nu$
Y _{57,11}	0	(5)				
Y _{59,3}	$\mathbf{m}_{57,3}$	(2)	0	$\mathbf{s}_{57,3}$	(1)	$\nu\Delta^2\nu^2$
Y _{60,10}	0	(1)				
Y _{60,12}	0	(1)				
Y _{62,2}	$\mathbf{m}_{60,12}$	(2)		$\mathbf{s}_{60,12}$	(1)	$\nu\Delta^2\nu^3$
Y _{65,7}	0	(1)				
Y _{65,13}	0	(1)				
Y _{66,2}	0	(1)				

Table 1: Detecting elements in \mathbf{tmf}_*

$\mathbf{img}(p_3)$	$\mathbf{img}(p_2)$	(T)	$\mathbf{img}(p_1)$	$i_1^{-1}(-)$	(T)	name in \mathbf{tmf}_*
Y68,2	$\mathbf{m}_{66,2}$	(2)	$\mathbf{s}_{65,3}$		(5)	$\eta\Delta\bar{\kappa}^2$
Y69,3	0	(1)				
Y70,8	0	(6)				
Y70,10	$\mathbf{m}_{68,10}$	(6)	0	$\mathbf{s}_{68,4}$	(1)	$4\Delta^2\bar{\kappa}$
Y71,3	$\mathbf{m}_{69,3}$	(5)	$\mathbf{s}_{68,4}$	0	(2)	$4\Delta^2\bar{\kappa}$
Y71,9	0	(5)				
Y74,4	$\mathbf{m}_{72,6}$	(2)	0	$\mathbf{s}_{72,0}$	(1)	$8\Delta^3$
Y75,13	0	(1)				
Y76,10	0	(1)				
Y77,5	$\mathbf{m}_{75,13}$	(2)	0	$\mathbf{s}_{75,3}$	(1)	$(\eta\Delta)^3$
Y80,16	0	(1)				
Y81,11	0	(1)				
Y82,6	$\mathbf{m}_{80,16}$	(2)	0	$\mathbf{s}_{80,16}$	(1)	$\bar{\kappa}^4$
Y83,3	$\mathbf{m}_{81,3}$	(2)	$\mathbf{s}_{80,16}$		(2)	$\bar{\kappa}^4$
Y85,17	0	(1)				
Y86,12	0	(1)				
Y87,7	$\mathbf{m}_{85,13}$	(2)	0	$\mathbf{s}_{85,13}$	(1)	$\eta\Delta\bar{\kappa}^3$
Y88,6	$\mathbf{m}_{86,12}$	(2)	$\mathbf{s}_{85,13}$		(2)	$\eta\Delta\bar{\kappa}^3$
Y90,14	0	(1)				
Y91,13	0	(1)				
Y92,8	$\mathbf{m}_{90,10}$	(2)	0	$\mathbf{s}_{90,10}$	(1)	$\eta^2\Delta^2\bar{\kappa}^2$
Y93,3	$\mathbf{m}_{91,9}$	(2)	$\mathbf{s}_{90,10}$		(2)	$\eta^2\Delta^2\bar{\kappa}^2$
Y96,14	0	(1)				
Y97,9	0	(1)				
Y98,4	$\mathbf{m}_{96,6}$	(2)	0	$\mathbf{s}_{96,0}$	(1)	$2\Delta^4$
Y101,15	0	(1)				
Y102,2	0	(3)				
Y102,10	$\mathbf{m}_{100,20}$	(3)	0	$\mathbf{s}_{100,20}$	(1)	$\bar{\kappa}^5$
Y103,7	$\mathbf{m}_{101,7}$	(4)	$\mathbf{s}_{100,20}$		(2)	$\bar{\kappa}^5$
Y105,21	0	(1)				
Y106,16	0	(1)				
Y107,3	$\mathbf{m}_{105,3}$	(5)	0	$\mathbf{s}_{105,3}$	(1,5)	$\nu^3\Delta^4$
Y107,11	$\mathbf{m}_{105,11}$	(5)	0	$\mathbf{s}_{105,17}$	(1,5)	$\eta\Delta\bar{\kappa}^4$
Y108,10	$\mathbf{m}_{106,16}$	(2)	$\mathbf{s}_{105,17}$		(2)	$\eta\Delta\bar{\kappa}^4$
Y111,17	0	(1)				
Y112,12	0	(1)				
Y113,7	$\mathbf{m}_{111,13}$	(2,5)	$\mathbf{s}_{110,14}$		(3)	$\eta^2\Delta^2\bar{\kappa}^3$
Y117,3	0	(1)				
Y117,13	0	(1)				
Y118,8	$\mathbf{m}_{116,10}$	(3)	0	$\mathbf{s}_{116,4}$	(1)	$2\Delta^4\bar{\kappa}$
Y119,3	$\mathbf{m}_{117,3}$	(2)	$\mathbf{s}_{116,4}$		(5)	$2\Delta^4 \cdot 2\bar{\kappa}$
Y122,4	$\mathbf{m}_{120,6}$	(5)	0	$\mathbf{s}_{120,0}$	(1)	$8\Delta^5$
Y122,14	0	(5)				

Table 1: Detecting elements in \mathbf{tmf}_*

$\text{img}(p_3)$	$\text{img}(p_2)$	(T)	$\text{img}(p_1)$	$i_1^{-1}(-)$	(T)	name in \mathbf{tmf}_*
$Y_{123,11}$	0	(1)				
$Y_{127,15}$	$\mathbf{m}_{125,21}$	(2)	0	$\mathbf{s}_{125,21}$	(5)	$\eta\Delta\bar{\kappa}^5$
$Y_{128,14}$	$\mathbf{m}_{126,20}$	(6)	$\mathbf{s}_{125,21}$		(2)	$\eta\Delta\bar{\kappa}^5$
$Y_{132,16}$	0	(1)				
$Y_{133,11}$	$\mathbf{m}_{131,17}$	(2,5)	$\mathbf{s}_{130,18}$		(2)	$\eta^2\Delta^2\bar{\kappa}^4$
$Y_{137,17}$	0	(1)				
$Y_{138,12}$	$\mathbf{m}_{136,14}$	(3)	0	$\mathbf{s}_{136,8}$	(3)	$\eta^2\Delta^5\kappa$
$y_{142,18}$	0	(1)				
$Y_{143,15}$	0	(1)				
$Y_{148,18}$	0	(1)				
$Y_{150,2}$	0	(4)				
$Y_{153,11}$	0	(3)				
$Y_{153,15}$	$\mathbf{m}_{151,21}$	(3)	$\mathbf{s}_{150,22}$		(2)	$\eta^2\Delta^2\bar{\kappa}^5$
$Y_{155,3}$	$\mathbf{m}_{153,3}$	(2)	0	$\mathbf{s}_{153,3}$	(1)	$\nu\Delta^6\nu^2$
$Y_{158,16}$	$\mathbf{m}_{156,18}$	(2)	0	$\mathbf{s}_{156,12}$	(3)	$\nu\Delta^6\eta\epsilon$
$Y_{161,7}$	0	(1)				
$Y_{165,3}$	0	(1)				
$Y_{167,3}$	$\mathbf{m}_{165,3}$	(2)	$\mathbf{s}_{164,4}$		(2)	$4\Delta^6\bar{\kappa}$
$Y_{168,22}$	0	(1)				
$Y_{170,4}$	$\mathbf{m}_{168,6}$	(2)	0	$\mathbf{s}_{168,0}$	(1)	$8\Delta^7$

3. NEW INFINITE FAMILIES

We begin by studying the commutative diagram of long exact sequences

$$(14) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & \pi_k Y & \xrightarrow{i_3} & \pi_k A_1 & \xrightarrow{p_3} & \pi_{k-3} Y \xrightarrow{v_*} \cdots \\ & & \downarrow h_{\mathbf{tmf}} & & \downarrow h_{\mathbf{tmf}} & & \downarrow h_{\mathbf{tmf}} \\ \cdots & \longrightarrow & \mathbf{tmf}_k Y & \xrightarrow{i_3} & \mathbf{tmf}_k A_1 & \xrightarrow{p_3} & \mathbf{tmf}_{k-3} Y \xrightarrow{v_*} \cdots \end{array}$$

associated to the cofiber sequence (4).

Lemma 3.1. Any nonzero element of the form $p_3(a) \in \mathbf{tmf}_* Y$ admits a lift in $\pi_* Y$ along the \mathbf{tmf} -Hurewicz homomorphism.

Proof. This is a straightforward consequence of the fact that the \mathbf{tmf} -Hurewicz map for A_1 (5) is a surjection [Pha23], along with the commutativity of (14). \square

Next, we study the commutative diagram of long exact sequences

$$(15) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & \pi_{k-3}M & \xrightarrow{i_2} & \pi_{k-3}Y & \xrightarrow{p_2} & \pi_{k-5}M & \xrightarrow{\eta_*} & \cdots \\ & & \downarrow \mathbf{h}_{\mathbf{tmf}} & & \downarrow \mathbf{h}_{\mathbf{tmf}} & & \downarrow \mathbf{h}_{\mathbf{tmf}} & & \\ \cdots & \longrightarrow & \mathbf{tmf}_{k-3}M & \xrightarrow{i_2} & \mathbf{tmf}_{k-3}Y & \xrightarrow{p_2} & \mathbf{tmf}_{k-5}M & \xrightarrow{\eta_*} & \cdots \end{array}$$

associated to the cofiber sequence (3).

Lemma 3.2. Any nonzero element of the form $p_2(p_3(a)) \in \mathbf{tmf}_*M$ admits a lift in π_*M along the \mathbf{tmf} -Hurewicz homomorphism.

Proof. If $p_2(p_3(a)) \neq 0$ then, in particular, $p_3(a) \neq 0$. Thus, by Lemma 3.1, there exists

$$\tilde{y} \neq 0 \in \pi_*Y$$

such that $\mathbf{h}_{\mathbf{tmf}}(\tilde{y}) = p_3(a)$. The result then follows from (15). \square

Remark 3.3. The action of Δ^8 is faithful on \mathbf{tmf}_*A_1 [Pha23], \mathbf{tmf}_*Y [BBPX22], \mathbf{tmf}_*M [BBPX22], \mathbf{tmf}_* [Bau08], the Hurewicz image of \mathbf{tmf}_* [BMQ23], and the cokernel of the \mathbf{tmf} -Hurewicz map [BMQ23].

3.1. Infinite families in 2-local stable stems.

Our final step is studying the commutative diagram of long exact sequences

$$(16) \quad \begin{array}{ccccccc} \cdots & \longrightarrow & \pi_{k-5}\mathbb{S} & \xrightarrow{i_1} & \pi_{k-5}M & \xrightarrow{p_1} & \pi_{k-6}\mathbb{S} & \xrightarrow{\cdot 2} & \cdots \\ & & \downarrow \mathbf{h}_{\mathbf{tmf}} & & \downarrow \mathbf{h}_{\mathbf{tmf}} & & \downarrow \mathbf{h}_{\mathbf{tmf}} & & \\ \cdots & \longrightarrow & \mathbf{tmf}_{k-5} & \xrightarrow{i_1} & \mathbf{tmf}_{k-5}M & \xrightarrow{p_1} & \mathbf{tmf}_{k-6} & \xrightarrow{\cdot 2} & \cdots \end{array}$$

associated to the cofiber sequence (2).

Suppose $p_1(p_2(p_3(a))) \neq 0$ for some $a \in \mathbf{tmf}_kA_1$. Then it follows from (14), Lemma 3.2, and Remark 3.3 that there is a 192-periodic infinite family

$$\{\tilde{\mathbf{s}}_{k-6+192i} \in \pi_{k-6+192i}(\mathbb{S}) : i \in \mathbb{N}\}$$

such that

- (1) $\mathbf{h}_{\mathbf{tmf}}(\tilde{\mathbf{s}}_{k-6}) = p_1(p_2(p_3(a)))$,
- (2) $\mathbf{h}_{\mathbf{tmf}}(\tilde{\mathbf{s}}_{k-6+192i}) \neq 0$ for all $i \in \mathbb{N}$.

We are interested in the case when $p_1(p_2(p_3(a))) = 0 \in \mathbf{tmf}_{k-6}$.

Theorem 3.4. Let $a \in \mathbf{tmf}_kA_1$ such that $p_2(p_3(a)) \neq 0$ and $p_1(p_2(p_3(a))) = 0$.

- (I) If $i_1^{-1}(p_2(p_3(a))) \cap \text{img}(\mathbf{h}_{\mathbf{tmf}}) \neq \emptyset$, then there exists a 192-periodic infinite family of elements in the stable stems

$$\{\tilde{\mathbf{s}}_{k-5+192i} \in \pi_{k-5+192i}(\mathbb{S}) : i \in \mathbb{N}\}$$

such that $i_1(\mathbf{h}_{\mathbf{tmf}}(\tilde{\mathfrak{s}}_{k-5})) = p_2(p_3(a))$ and $\mathbf{h}_{\mathbf{tmf}}(\tilde{\mathfrak{s}}_{k-5+192i}) \neq 0$ for all $i \in \mathbb{N}$.

(II) If $i_1^{-1}(p_2(p_3(a))) \cap \text{img}(\mathbf{h}_{\mathbf{tmf}}) = \emptyset$, then there exists a 192-periodic infinite family of elements in the stable stems

$$\{\tilde{\mathfrak{s}}_{k-6+192i} \in \pi_{k-6+192i}(\mathbb{S}) : i \in \mathbb{N}\}$$

such that $\tilde{\mathfrak{s}}_{k-6+192i} \neq 0$ and $\mathbf{h}_{\mathbf{tmf}}(\tilde{\mathfrak{s}}_{k-6+192i}) = 0$ for all $i \in \mathbb{N}$.

Proof. If $p_2(p_3(a)) \in \mathbf{tmf}_{k-5}\mathbf{M}$ is nonzero, then by Remark 3.3,

$$p_2(p_3(\Delta^{8i} \cdot a_k)) = \Delta^{8i} \cdot p_2(p_3(a_k)) \neq 0.$$

Thus, by Lemma 3.2, there exist nonzero elements

$$(17) \quad \tilde{m}_{k-5+192i} \in \pi_{k-5+192i}(\mathbf{M})$$

such that $\mathbf{h}_{\mathbf{tmf}}(\tilde{m}_{k-5+192i}) = \Delta^{8i} \cdot p_2(p_3(a_k))$ for all $i \in \mathbb{N}$.

Suppose $p_2(p_3(a))$ admits a lift $s_{k-5} \in \mathbf{tmf}_{k-5}$ along i_1 which is in the Hurewicz image. Then, by Remark 3.3, $\Delta^{8i} \cdot s_{k-5}$ is also in the Hurewicz image, and a collection

$$\{\tilde{\mathfrak{s}}_{k-5+192i} \in \pi_{k-5+192i}(\mathbb{S}) : i \in \mathbb{N}\}$$

such that $\mathbf{h}_{\mathbf{tmf}}(\tilde{\mathfrak{s}}_{k-5+192i}) = \Delta^{8i} \cdot s_{k-5}$ forms an infinite family with the desired properties.

On the other hand, if none of the lifts of $p_2(p_3(a))$ along i_1 is in the Hurewicz image, then the same holds for $\Delta^{8i} \cdot p_2(p_3(a))$ for all $i \in \mathbb{N}$ by Remark 3.3. Thus, $p_1(\tilde{m}_{k-5+192i}) \neq 0$ for all $i \in \mathbb{N}$, and

$$\{\tilde{\mathfrak{s}}_{k-6+192i} = p_2(\tilde{m}_{k-5+192i}) : i \in \mathbb{N}\}$$

is the desired infinite family. \square

Proof of Theorem 1. From Table 1 we notice that there exists an element $a_k \in \mathbf{tmf}_k A_1$ such that

$$(i) \quad p_2(p_3(a_k)) \neq 0,$$

$$(ii) \quad p_1(p_2(p_3(a_k))) = 0,$$

$$(iii) \quad \text{there exists } s_{k-5} \in \mathbf{tmf}_{k-5}^{\text{tor}} \text{ such that } i_1(s_{k-5}) = p_2(p_3(a_k)),$$

for each $k \in \{29, 53, 77, 80, 101, 119, 173\}$. However, the Hurewicz image is trivial in degrees 24, 48, 72, 75, 96, 114, and 168. Thus, the result follows from Case (II) of Theorem 3.4. \square

Remark 3.5. To summarize, the seven infinite families in Theorem 1 are a consequence of the fact that the elements

$$(18) \quad 8\Delta, 4\Delta^2, 8\Delta^3, (\eta\Delta)^3, 2\Delta^4, 8\Delta^5, 8\Delta^7$$

which are not in the Hurewicz image of \mathbf{tmf}_* are the lifts of nonzero elements in the image of $p_2 \circ p_1 : \mathbf{tmf}_* A_1 \rightarrow \mathbf{tmf}_{*-5}\mathbf{M}$ along i_1 .

3.2. Infinite families in $K(2)$ -local stable stems.

Theorem 3.6. *All elements listed in Theorem 1 have nonzero images in the $K(2)$ -local stable stems.*

Notation 3.7. For a finite spectrum X , let \widehat{X} denote its $K(2)$ -localization.

The work in [Pha23] shows that $K(2)$ -local Hurewicz map of A_1

$$h_{\mathrm{TMF}} : \pi_* \widehat{A}_1 \longrightarrow \mathrm{TMF}_* A_1$$

is a surjection.

Since $\mathrm{TMF}_* A_1 \cong (\Delta^8)^{-1} \mathrm{tmf}_* A_1$ and the action of Δ^8 on $\mathrm{tmf}_* A_1$ is faithful (see Remark 3.3), the natural map

$$\ell : \mathrm{tmf}_* A_1 \longrightarrow \mathrm{TMF}_* A_1$$

is an injection. Thus the image of $\ell(a) \in \mathrm{TMF}_* \widehat{A}_1$ under the map

$$p_2 \circ p_3 : \mathrm{TMF}_* A_1 \longrightarrow \mathrm{TMF}_* M$$

is nonzero if and only if $p_2(p_3(a)) \in \mathrm{tmf}_* M$ is nonzero. Similarly, the image of $\ell(a) \in \mathrm{TMF}_* \widehat{A}_1$ under the map

$$p_1 \circ p_2 \circ p_3 : \mathrm{TMF}_* \widehat{A}_1 \longrightarrow \mathrm{TMF}_{*-6}$$

is zero if and only if $p_1(p_2(p_3(a))) \in \mathrm{TMF}_{*-6}$. Therefore, the proof of Theorem 3.6 can follow the exact same arguments to that of Theorem 1 provided

$$8\Delta, 4\Delta^2, 8\Delta^3, (\eta\Delta)^3, 2\Delta^4, 8\Delta^5, 8\Delta^7$$

are not in the $K(2)$ -local Hurewicz image of TMF_* (also see Remark 3.5).

Proof of Theorem 3.6. Since the elements

$$8\Delta, 4\Delta^2, 8\Delta^3, 2\Delta^4, 8\Delta^5, 8\Delta^7$$

are integral classes and $\pi_* \widehat{\mathbb{S}}$ is a finite group in degrees 24, 48, 72, 96, 120, and 168 (see [BSSW24, Theorem A]), they cannot be in the $K(2)$ -local Hurewicz image. Further, $(\eta\Delta)^3$ is also not in the Hurewicz image by Lemma 3.8. Hence, the result. \square

Lemma 3.8. The element $(\eta\Delta)^3$ is not in the image of the $K(2)$ -local Hurewicz map of TMF

$$(19) \quad h_{\mathrm{TMF}} : \pi_* \widehat{\mathbb{S}} \longrightarrow \mathrm{TMF}_*.$$

Proof. By [Lau04, Corollary 3]

$$v_1^{-1} \mathrm{TMF} \simeq \mathrm{KO}[j^\pm],$$

which implies that the $K(2)$ -local Hurewicz map of $v_1^{-1}\mathrm{TMF}$ factors through KO_*

$$\begin{array}{ccccc} & & \mathrm{KO}_* & & \\ & \nearrow & & \searrow & \\ \pi_*\widehat{\mathbb{S}} & \longrightarrow & \pi_*\mathrm{TMF} & \longrightarrow & v_1^{-1}\mathrm{TMF}_* \end{array}$$

Then we simply implement the arguments of [BMQ23, Theorem 6.1].

More precisely, we observe that $(\eta\Delta)^3$ lifts to an element in $\mathrm{TMF}_*\mathrm{M}(\infty)$, where $\mathrm{M}(\infty) := \operatorname{colim}_{i \rightarrow \infty} \mathrm{M}(i)$ (see Notation 3.11), whose image after inverting c_4 is

$$\overline{v_1^{38}j^{-3}} \in v_1^{-1}\mathrm{TMF}_*\mathrm{M}(\infty)$$

in the notations of [BMQ23, §6]. If $(\eta\Delta)^3$ is in the image of the Hurewicz map (19), then $\overline{v_1^{38}j^{-3}}$ must also be in the image of $\ell \circ h_{\mathrm{TMF}}$ in the diagram

$$\begin{array}{ccccc} & & \mathrm{KO}_*\mathrm{M}(\infty) & & \\ & \nearrow & & \searrow & \\ \pi_*\widehat{\mathrm{M}}(\infty) & \xrightarrow{h_{\mathrm{TMF}}} & \mathrm{TMF}_*\mathrm{M}(\infty) & \xrightarrow{\ell} & v_1^{-1}\mathrm{TMF}_*\mathrm{M}(\infty) \end{array}$$

which contradicts the fact that $\ell \circ h_{\mathrm{TMF}}$ factors through $\mathrm{KO}_*\mathrm{M}(2^\infty)$. \square

Since the unit map of $\widehat{\mathbb{S}}$ factors through that of $\mathbb{S}_{T(2)}$ (see (1)), the proof of Theorem 3.6 completes the proof of Theorem 2. Nevertheless, we take this opportunity to give an independent proof of the fact that the elements listed in Theorem 1 are nontrivial after $T(2)$ -localization.

3.3. Infinite families in $T(2)$ -local stable stems.

It is well-known that A_1 admits a v_2^{32} -self-map

$$v : \Sigma^{192}A_1 \longrightarrow A_1$$

detected by $\Delta^8 \in \mathbf{tmf}_*$ [BEM17]. Therefore, for any lift $\tilde{a} \in \pi_k A_1$ of $a \in \mathbf{tmf}_k(A_1)$ we have

$$h_{\mathbf{tmf}}(v_2^{32i} \cdot \tilde{a}) = \Delta^{8i} \cdot a$$

for all $i \in \mathbb{N}$.

Notation 3.9. For a spectrum E and a finite spectrum X with a v_n -self-map $v : \Sigma^{|v|}X \rightarrow X$, let

$$\Phi_X(E) := \operatorname{colim}_{\rightarrow} \{E^X \xrightarrow{v^*} \Sigma^{-|v|}E^X \xrightarrow{v^*} \Sigma^{-2|v|}E^X \longrightarrow \dots\}.$$

There is a natural map from E to $\Phi_X(E)$ which we will denote by α (sometimes with subscripts).

Suppose $\tilde{a} \in \pi_k A_1$ such that $\mathfrak{p}_*(\tilde{a}) \in \pi_{k-6}(\mathbb{S})$ is listed in [Theorem 1](#), then we can choose $\tilde{m}_{k-5+192i}$ of [\(17\)](#) as

$$\tilde{m}_{k-5+192i} := p_2(p_3(v_2^{32i} \cdot \tilde{a}))$$

for all $i \in \mathbb{N}$ in the proof of [Theorem 3.4 \(II\)](#). As a result, we conclude:

Lemma 3.10. Suppose $\tilde{a} \in \pi_k A_1$ such that $\mathfrak{p}_*(\tilde{a}) \in \pi_{k-6}(\mathbb{S})$ is listed in [Theorem 1](#). Then the image \tilde{a} under the map

$$\alpha_1 : \pi_* \mathbb{S} \longrightarrow \pi_*(\Phi_{A_1}(\mathbb{S}))$$

is nonzero.

Notation 3.11. Let $M(i, j)$ denote the cofiber of a v_1^j -self-map on $M(i)$, the cofiber of multiplication by 2^i on \mathbb{S} .

Proposition 3.12. The map $\mathfrak{p} : \Sigma^{-6} A_1 \longrightarrow \mathbb{S}$ factors through $M(1, 4)$.

Proof. Let \mathfrak{ko} denote the connective real K-theory. Since $\mathfrak{ko}_6 M \cong 0$ it follows that the composite

$$\Sigma^6 \mathbb{S} \hookrightarrow \Sigma^6 Y \xrightarrow{v_1^3} Y \xrightarrow{\mathfrak{p}_2} \Sigma^2 M$$

is nonzero in \mathfrak{ko} -homology, and hence, in stable homotopy. Further, we have a commutative diagram

$$\begin{array}{ccc} \Sigma^6 Y & \xrightarrow{v_1} & \Sigma^4 Y \\ \mathfrak{p}_2 \downarrow & & \downarrow \mathfrak{p}_2 \circ v_1^3 \\ \Sigma^8 \mathbb{S}/2 & \xrightarrow{v_1^4} & \mathbb{S}/2 \end{array}$$

which implies that there is a map $\Sigma^4 A_1 \longrightarrow M(1, 4)$ which factors the pinch map of $\Sigma^4 A_1$ to its top cell. \square

A consequence of [Proposition 3.12](#) is that we have a directed system

$$(20) \quad \Sigma^{-6} A_1 \rightarrow M(1, 4) \rightarrow \Sigma^{-18} M(2, 8) \rightarrow \dots$$

of type 2 spectra which is cofinal among all type 2 spectra with a ‘pinch’ map to \mathbb{S} .

Notation 3.13. Let $\Phi_k(-)$ denote $\Phi_{V_k}(-)$, where V_k is the k -th entry of the sequence [\(20\)](#).

Theorem 3.14. All elements listed in [Theorem 1](#) have nonzero images in the $T(2)$ -local stable stems.

Proof. We will make use of the standard theory of Bousfield-Kuhn functors (see [\[Kuh08\]](#)) which implies

$$\mathbb{S}_{T(2)} \simeq \varprojlim \Phi_k(\mathbb{S}).$$

Since, the image under α_1 of an element $\tilde{a} \in \pi_*(A_1)$ listed in [Theorem 1](#) is nonzero, the diagram

$$\begin{array}{ccccc}
 \pi_*(\mathbb{S}) & & & & \\
 \downarrow \alpha_1 & \searrow \alpha_2 & & \searrow \dots & \\
 \pi_*(\Phi_1(\mathbb{S})) & \longleftarrow & \pi_*(\Phi_2(\mathbb{S})) & \longleftarrow & \dots
 \end{array}$$

implies that the image of \tilde{a} in $\lim_{\leftarrow} \pi_* \Phi_k(\mathbb{S})$ is nonzero. Then the result follows from the fact that the natural map

$$\pi_*(\mathbb{S}_{T(2)}) \cong \pi_*\left(\lim_{\leftarrow} \mathbb{S}_{T(2)}\right) \longrightarrow \lim_{\leftarrow} \pi_* \Phi_k(\mathbb{S})$$

is a surjection (with Milnor \lim^1 term as the kernel). □

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