Bailey 4-23-10

Note Title 4/23/2010

Tangent schemes + derivations

Tangent scheme

Let A = comm ring

Dual numbers $A(\varepsilon) = A[\varepsilon]/\varepsilon^2$

Let $X : Alg_{\mathbb{R}} \longrightarrow$ Sets be a scheme

Define the tangent functor Tan_X

$$Tan_X(A) = X(A(\varepsilon))$$

$q : A(\varepsilon) \longrightarrow A$ augmentation
$q(\varepsilon) = 0$

$X(A(\varepsilon)) \longrightarrow X(A)$

$\mathrm{Tan}_X \longrightarrow X$

There is also a unit map $u : A \longrightarrow A(\varepsilon)$

$X(A) \longrightarrow X(A(\varepsilon))$

$s_0 : X \longrightarrow \mathrm{Tan}_X$ is the <u>zero section</u>

If $f : X \longrightarrow S$ is a morphism of R-functors

$X(A(\varepsilon)) \longrightarrow S(A(\varepsilon))$

$\mathrm{Tan}_X \longrightarrow \mathrm{Tan}_S$

$$\begin{array}{ccc} \mathrm{Tan}_{X/S} & \longrightarrow & \mathrm{Tan}_X \\ \downarrow & \text{pullback} & \downarrow \\ S & \xrightarrow[s_0]{} & \mathrm{Tan}_S \end{array}$$

$\mathrm{Tan}_{X/S}$ is the <u>tangent functor of X relative to S</u>.

Let Mod_A be the category of module sheaves over $\mathcal{O}_A$ in the Zariski topology.

$\mathcal{O}_A$ is the constant sheaf sending every open set to A.

If $\mathcal{F}$ is a quasi-coherent sheaf

$$\mathcal{O}_X^{(I)}(U) \longrightarrow \mathcal{O}_X^{(J)}(U) \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

is an exact sequence for each open U of $\mathcal{O}_X$-modules on a scheme X

X is covered by affine schemes

$$\mathrm{Spec}(B) \to X$$

$$\mathcal{O}_X(\mathrm{Spec}(B) \to X) = B$$

Define a set-valued functor on R-algebras A by

$$\mathbb{V}(\mathcal{F})(A) = \coprod_{\mathrm{Spec}(A) \to X} \mathrm{Mod}_A(\mathcal{F}(A), A)$$

Prop If $X \to S$ is a separated morphism of schemes, there is a natural iso of abelian gp schemes with

$$V(\Omega_{X/S}) \cong Tan_{X/S}$$

Terms to be explained later

$\Omega_{X/S}$ = derivations of X over S.

Derivations + differentials (classical)

If B is an R-algebra, an R-derivation is a morphism $d: B \to M$ (a B-module)

- $d(b_1 + b_2) = db_1 + db_2$
- $d(b_1 b_2) = (db_1)\, b_2 + b_1 (db_2)$
- $d(r) = 0$ for $r \in R$

The module of differentials $\Omega_{B/R}$ is a B-module with a derivation. $d: B \to \Omega_{B/R}$ such that any other derivation factors thru it

$$\begin{tikzcd} B \arrow[rr, "d_M"] \arrow[dr, "d"'] & & M \\ & \Omega_{B/R} \arrow[ur, dashed] & \end{tikzcd}$$

<u>How to construct $\Omega_{B/R}$</u>. Consider the free B-module $D = \{db \mid b \in B\}$

$$\Omega_{B/R} = D \,/\, \left(\text{relations forced by definition of derivation}\right)$$

Example $B \otimes_R B \xrightarrow{\mu} B$

$$\mu(b_1 \otimes b_2) = b_1 b_2$$

Define $J = \ker \mu$

$J < B \otimes_R B$ also has left B-structure induced by left B action on $B \otimes_R B$.

$$d_J : B \longrightarrow J$$

$$d_J(b_1) = b_1 \otimes 1 - 1 \otimes b$$

$$d_J(b_1 + b_2) = d_J b_1 + d_J b_2$$

$$d_J(b_1 b_2) = b_1 b_2 \otimes 1 + b_1 \otimes b_2 - b_1 \otimes b_2 - 1 \otimes b_1 b_2$$

$$= b_1 d_J(b_2) - (d_J b_1) b_2$$

$b_1 \otimes b_2 \in J^2$ so $d_J : B \longrightarrow J/J^2$

Most books define $\Omega_{B/R} = J/J^2$ which is our def for certain B (?).

Def Let A be a ring and M an A-module. There is a sheaf $\tilde{M}$ over $\mathrm{Spec}(A)$ associated to M.

M localized a $\mathfrak{p} \subset A$

$$\tilde{M}(U) = \left\{ s : U \to \coprod_{\mathfrak{p} \in U} M_{\mathfrak{p}} \;\middle|\; s(\mathfrak{p}) \in M_{\mathfrak{p}} \right\}$$

$s(\mathfrak{q}) = m/f$

$m \in M_{\mathfrak{q}}$ $\quad f \notin \mathfrak{q}$.

Prop

$X = \mathrm{Spec}$

① $\tilde{M}$ is an $\mathcal{O}_X$-module (i.e. an A-module)

② For each $\mathfrak{p} \in \mathrm{Spec}(A)$ the stalk of $(\tilde{M})_{\mathfrak{p}}$

③ $\Gamma(\overset{\text{is}}{\mathrm{Spec}(A)}, \overset{M_p}{\tilde{M}}) = M$

Def. A morphism of schemes $X \xrightarrow{f} Y$ is *separated* if

$$\begin{tikzcd} X \times_Y X \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pull back}"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

$\Delta : X \to X \times_Y X$ is a closed immersion

Def Suppose Y is a closed subscheme of X with inclusion $i : Y \to X$. Then the *ideal sheaf $\mathcal{J}_Y$ of Y* is the kernel of

$$i^\# : \mathcal{O}_X \to i_* \mathcal{O}_Y$$

Note There is a 1-1 correspondence between closed subschemes between

$$\left\{ \begin{array}{l} Y \subset X \\ \text{closed subscheme} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{quasi-coherent} \\ \text{sheaves over } X \end{array} \right\}$$

$$Y \longmapsto \mathcal{J}_Y$$

$$Y = \operatorname{supp} \mathcal{O}_X / \mathcal{J} \longleftarrow\!\mapsto \mathcal{J}$$

Example

Suppose there is a separated scheme morphism $f : X \to Y$ (Recall $\Delta(X) \subset X \times_Y X$ is closed). Let

Let $U = \mathrm{Spec}(R)$ be an open subset of Y

Let $V = \mathrm{Spec}(B)$ " X

such that $f(V) \subset U$

$$\begin{tikzcd} V \times_U V \arrow[r] \arrow[d] & V \arrow[d, "f"] \\ V \arrow[r, "f"] & U \end{tikzcd}$$

Note $\mathrm{Spec}(B) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(B)$

$\|$

$\mathrm{Spec}(B \otimes_R B)$

$\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(B \otimes_R B)$

$B \longleftarrow B \otimes_R B$

$\Delta(X) \cap (V \times_U V)$ is a closed subscheme of $X \times_Y X$ corresponding to the ideal sheaf $\mathcal{J}/\mathcal{J}^2 \Longleftrightarrow J/J^2$.

$$\Omega_{U/V} = (\Omega_{B/R})^{\sim}$$

We can glue these together and: $B \longrightarrow \Omega_{B/R}$ to get a derivation

$\Omega_{X/Y}$

$$d: \mathcal{O}_X \longrightarrow \mathcal{O}_{X/Y}$$

$$\Omega_{X/Y} \cong \Delta^*(\mathcal{J}/\mathcal{J}^2)$$