MULTIPLICATIVITY OF THE IDEMPOTENT SPLITTINGS OF THE BURNSIDE RING AND THE G-SPHERE SPECTRUM

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Abstract. We provide a complete characterization of the equivariant commutative ring structures of all the factors in the idempotent splitting of the G-equivariant sphere spectrum, including their Hill-Hopkins-Ravenel norms, where G is any finite group. Our results describe explicitly how these structures depend on the subgroup lattice and conjugation in G. Algebraically, our analysis characterizes the multiplicative transfers on the localization of the Burnside ring of G at any idempotent element, which is of independent interest to group theorists. As an application, we obtain an explicit description of the incomplete sets of norm functors which are present in the idempotent splitting of the equivariant stable homotopy category.

1. Introduction

Let G be a finite group and recall that the zeroth G-equivariant homotopy group $\pi_0^G(\mathbb{S})$ of the G-sphere spectrum identifies with the Burnside ring $A(G)$ [Seg71]. Dress' classification [Dre69] of the primitive idempotent elements $e_L \in A(G)$ in terms of perfect subgroups $L \leq G$ gives rise to a splitting of G-spectra

$$\mathbb{S} \simeq \prod_{(L) \leq G} \mathbb{S}[e_L^{-1}] \tag{1.1}$$

where the localization $\mathbb{S}[e_L^{-1}]$ is the sequential homotopy colimit

$$\operatorname{hocolim}(\mathbb{S} \xrightarrow{e_L} \mathbb{S} \xrightarrow{e_L} \ldots)$$

along countably many copies of (a representative of) e_L. The present paper investigates the multiplicative nature of this splitting.

The sphere is a commutative monoid in any good symmetric monoidal category of G-spectra and hence admits the structure of a G-E_∞ *ring spectrum*, i.e., it comes equipped

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with a full set of *Hill-Hopkins-Ravenel norm maps*

$$N_K^H\colon \bigwedge_{H/K} \mathrm{Res}_K \mathbb{S} \to \mathrm{Res}_H \mathbb{S}$$

for all $K \leq H \leq G$. These are equivariantly commutative multiplication maps which feature prominently in the solution to the Kervaire invariant problem [HHR16]. The resulting norms on homotopy groups first appeared in [GM97]. They are multiplicative transfer maps

$$N_K^H\colon \pi_0^K(\mathbb{S}) \cong A(K) \to A(H) \cong \pi_0^H(\mathbb{S})$$

which equip $\underline{\pi}_0(\mathbb{S}) \cong A(-)$ with the structure of a *Tambara functor* [Tam93] (and agree with the multiplicative transfers of $A(-)$ induced by co-induction of finite G-sets, see Section 3).

It is known that norm maps behave badly with respect to Bousfield localization of spectra and levelwise localization of Tambara functors, see Example 2.23. Thus, it is natural to ask about the equivariant multiplicative behavior of the idempotent splitting (1.1). Throughout the paper, we will decorate the norms of a localization with a tilde to distinguish them from the norms of the original object.

Question 1.2 (Main question, homotopy-theoretic formulation)**.** For which nested subgroups $K \leq H \leq G$ does the norm map N_K^H of $\mathbb{S}$ descend to a norm map

$$\tilde{N}_K^H\colon \bigwedge_{H/K} \mathrm{Res}_K \mathbb{S}[e_L^{-1}] \to \mathrm{Res}_H \mathbb{S}[e_L^{-1}]$$

on the idempotent localization $\mathbb{S}[e_L^{-1}]$, and which norms are preserved by the idempotent splitting (1.1)?

Question 1.3 (Main question, algebraic formulation)**.** For which nested subgroups $K \leq H \leq G$ does the Green ring $\underline{\pi}_0\mathbb{S}[e_L^{-1}] \cong A(-)[e_L^{-1}]$ inherit a norm map $\tilde{N}_K^H$ from that of $A(-)$, and which norms are preserved by the idempotent splitting

$$A(-) \cong \prod_{(L)\leq G} A(-)[e_L^{-1}]?$$

We now state our main results which provide an explicit and exhaustive answer to both questions. All of our results hold locally for any collection of primes inverted. For simplicity, we only include the integral statements in the introduction.

1.1. **Statement of algebraic results.**

The following result will be restated as Theorem 4.1, including the local variants.

Theorem A. *Let $L \leq G$ be a perfect subgroup and let $e_L \in A(G)$ be the corresponding primitive idempotent given by Dress' classification of idempotents in $A(G)$ (see Theorem 3.4). Fix subgroups $K \leq H \leq G$. Then the norm map $N_K^H \colon A(K) \to A(H)$ descends to a well-defined map of multiplicative monoids*

$$\tilde{N}_K^H \colon A(K)[e^{-1}] \to A(H)[e^{-1}]$$

if and only if one of the following holds:

i) Neither K nor H are super-conjugate in G to L.
ii) Both K and H are super-conjugate in G to L and satisfy the following condition:
($\star$) If $L' \leq G$ is conjugate in G to L and is contained in H, then it is contained in K.

Theorem A builds on previous work by Hill-Hopkins [HH14] and Blumberg-Hill [BH16, Section 5.4] which reduced the question to understanding certain division relations between norms and restrictions of the elements $e_L \in \pi_0^G(\mathbb{S})$, but did not make explicit the relationship with the subgroup structure of G. The proof of Theorem A is entirely algebraic and can be found in Section 4.2.

We now record some immediate consequences of Theorem A that will be restated as Corollary 4.2 and Corollary 4.3.

Corollary B. *Assume that $L \leq G$ is normal and perfect. Fix $K \leq H \leq G$. Then $A(-)[e_L^{-1}]$ inherits a norm $\tilde{N}_K^H$ from $A(-)$ if and only if one of the following two conditions is satisfied:*

i) Neither K nor H contain any G-conjugate of L as a subgroup.
ii) Both K and H contain some G-conjugate of L as a subgroup.

Conversely, if for some perfect $L \leq G$ the two conditions are equivalent to the existence of $\tilde{N}_K^H$, then L is necessarily normal in G.

Corollary C. *The Green ring $A(-)[e_L^{-1}]$ admits all norms $\tilde{N}_K^H$ for all $K \leq H$ if and only if $L = 1$ is the trivial group. In this case, the norm maps equip $A(-)[e_L^{-1}]$ with the structure of a Tambara functor.*

For an arbitrary perfect subgroup $L \leq G$, we explain how the levelwise localization $A(-)[e_L^{-1}]$ fits into Blumberg-Hill's framework of *incomplete Tambara functors* [BH16], the basics of which we recall in Section 2.3. For $K \leq H \leq G$, call the H-set H/K *admissible* for e_L if $K \leq H$ satisfy the conditions of Theorem A. Call a finite H-set admissible if all of its orbits are admissible. Theorem A is complemented by the following two structural results.

Theorem D (see Theorem 4.20)**.** *Let $L \leq G$ be a perfect subgroup and let $e_L \in A(G)$ be the corresponding primitive idempotent. Then the following hold:*

- *i) The admissible sets assemble into an indexing system $\mathcal{I}_L$ (in the sense of* [BH16, Def. 1.2], *see Section 2.1) such that $A(-)[e_L^{-1}]$ is an $\mathcal{I}_L$-Tambara functor under $A(-)$.*
- *ii) In the poset of indexing systems, $\mathcal{I}_L$ is maximal among the elements that satisfy i).*
- *iii) The map $A(-) \to A(-)[e_L^{-1}]$ is a localization at e_L in the category of $\mathcal{I}_L$-Tambara functors.*

Corollary E (see Corollary 4.24)**.** *The localization maps $A(-) \to A(-)[e_L^{-1}]$ assemble into a canonical isomorphism of $\mathcal{I}$-Tambara functors*

$$A(-) \to \prod_{(L) \leq G \text{ perfect}} A(-)[e_L^{-1}],$$

where $\mathcal{I}$ is the intersection

$$\mathcal{I} = \bigcap_{(L) \leq G} \mathcal{I}_L$$

of the indexing systems given by Theorem D.

Together, Theorem A, Theorem D and Corollary E answer Question 1.3. A simple characterization of the norms parametrized by $\mathcal{I}$ can be found in Lemma 4.23.

1.2. **Statement of homotopical results.** It was conjectured by Blumberg-Hill [BH15b, Section 5.2] and proven in [GW17, Rub17, BP17] that any indexing system can be realized by an N_∞ operad which encodes norms precisely for the admissible sets of that indexing system. In particular, for any of the indexing systems $\mathcal{I}_L$ of Theorem D, we can choose a corresponding Σ-cofibrant N_∞ operad $\mathcal{O}_L$. See Section 2.2 for details.

We use general preservation results for N_∞ algebras under localization [HH14, GW17] to lift our algebraic results about $\mathcal{I}_L$-Tambara functor structures on homotopy groups to a homotopical statement about $\mathcal{O}_L$-algebra structures on G-spectra. The following result is restated as Corollary 4.26.

Corollary F. *Let $\mathcal{O}_L$ be any Σ-cofibrant N_∞ operad whose associated indexing system is $\mathcal{I}_L$. Then:*

- *i) The G-spectrum $\mathbb{S}[e_L^{-1}]$ is (canonically equivalent to) an $\mathcal{O}_L$-algebra under $\mathbb{S}$.*
- *ii) In the poset of homotopy types of N_∞ operads, $\mathcal{O}_L$ is maximal among the elements that satisfy i).*
- *iii) The map $\mathbb{S} \to \mathbb{S}[e_L^{-1}]$ is a localization at e_L in the category of $\mathcal{O}_L$-algebras.*

A homotopical reformulation of Corollary C shows that the idempotent splitting of $\mathbb{S}$ is far from being a splitting of G-E_∞ ring spectra.

Corollary G (see Corollary 4.27)**.** *The G-spectrum $\mathbb{S}[e_L^{-1}]$ is a G-E_∞ ring spectrum if and only if $L = 1$ is the trivial group.*

Locally at a prime p, this recovers a (currently unpublished) result of Grodal [Gro, Cor. 5.5], which we state as Theorem 4.28.

There is a homotopical analogue of Corollary E.

Corollary H (see Corollary 4.29)**.** *Let $\mathcal{O}$ be any Σ-cofibrant N_∞ operad whose associated indexing system is $\mathcal{I}$. Up to equivalence of G-spectra, the idempotent splitting*

$$\mathbb{S} \simeq \prod_{(L)\leq G} \mathbb{S}[e_L^{-1}]$$

is an equivalence of $\mathcal{O}$-algebras, where the product is taken over conjugacy classes of perfect subgroups.

Together, Corollary F and Corollary H answer Question 1.2.

1.3. **Examples.** In Section 5, we use our results to explicitly calculate the multiplicative structure of the idempotent splittings of the sphere in the case of the alternating group A_5 and the symmetric group Σ_3 (working 3-locally). Moreover, for arbitrary G, we deduce that the rational idempotent splitting of $\mathbb{S}_\mathbb{Q}$ cannot preserve any non-trivial norm maps. The latter is not a new insight, cf. e.g. [BGK17, Section 7].

1.4. **Applications to modules.** Corollary F, together with the theory of modules of [BH15a], also characterizes the norm functors which arise on the level of modules over the N_∞ ring $\mathbb{S}[e_L^{-1}]$ and its restrictions to subgroups.

The following result will be restated as Corollary 6.1.

Corollary 1.4. *Let $L \leq G$ be perfect and let $\mathcal{O}_L$ as in Corollary F. Assume furthermore that $\mathcal{O}_L$ has the homotopy type of the linear isometries operad on some (possibly incomplete) universe U. For all admissible sets H/K of $\mathcal{I}_L$, there are norm functors relative to $\mathbb{S}[e_L^{-1}]$*

$$_{\mathrm{Res}_H(\mathbb{S}[e_L^{-1}])}N_{K,\mathrm{Res}_K(U)}^{H,\mathrm{Res}_H(U)} \colon \mathit{Mod}(\mathrm{Res}_K^G(\mathbb{S}[e_L^{-1}])) \to \mathit{Mod}(\mathrm{Res}_H^G(\mathbb{S}[e_L^{-1}]))$$

which on the level of homotopy categories form the structure of an "incomplete Mackey functor in symmetric monoidal categories".

Any G-spectrum is a module over $\mathbb{S}$, hence the idempotent splitting (1.1) of $\mathbb{S}$ induces a splitting of the category of G-spectra. Corollary 1.4 then says that this does not give rise to a splitting of G-symmetric monoidal categories in the sense of [HH16]. Indeed, the categories of modules over (restrictions to subgroups of) $\mathbb{S}[e_L^{-1}]$ will only admit an incomplete set of norm functors, which then can be read off from Theorem A.

1.5. **Future directions.** It would be interesting to study the idempotent splittings of other equivariant commutative ring spectra such as complex K-theory KU_G. However, the generalization of our results does not appear to be straightforward, as our approach relies on properties of the Burnside ring that break down in the case of the complex representation ring $RU(G) \cong \pi_0^G(KU_G)$, see Section 6.

Organization: Section 2 provides some background material on N_∞ operads and their algebras in G-spectra, (incomplete) Tambara functors, indexing systems and their behavior under localization. In Section 3, we recall Dress' classification of idempotent elements in the Burnside ring and explain how to obtain the splitting (1.1) of the G-equivariant sphere spectrum. We state and prove our results (including the local variants) in Section 4 and discuss examples in Section 5. Finally, applications and future directions are discussed in Section 6.

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2. Preliminaries

We briefly recall some background material on N_∞ operads and N_∞ ring spectra, incomplete Tambara functors and localizations. Most of this section follows [BH15b, BH16].

2.1. N_∞ **operads and indexing systems.** Recall that a subgroup $\Gamma \leq G \times \Sigma_n$ is a *graph subgroup* if it is the graph of a group homomorphism $H \to \Sigma_n$ for some $H \leq G$, or equivalently, if $\Gamma \cap (\{1\} \times \Sigma_n)$ is trivial. By a *G-operad* we mean an operad in the category of (unbased) G-spaces.

Definition 2.1 ([BH15b], Def. 1.1)**.** A G-operad $\mathcal{O}$ is called an N_∞ *operad* if each G-space $\mathcal{O}(n)$ is a universal space for a family $\mathcal{F}_n$ of graph subgroups of $G \times \Sigma_n$ which contains all graphs of trivial homomorphisms, i.e., all subgroups of the form $H \times \{\mathrm{id}\}$.

The following properties are immediate from the definition.

Lemma 2.2. *For an N_∞ operad $\mathcal{O}$, the following holds:*

(i) The G-spaces $\mathcal{O}(0)$ and $\mathcal{O}(1)$ are G-equivariantly contractible.
(ii) The action of Σ_n on $\mathcal{O}(n)$ is free.
(iii) The underlying non-equivariant operad is always an E_∞ operad.

Example 2.3 ([BH15b], Lemma 3.15)**.** Let U be a (not necessarily complete) G-universe, and let $\mathcal{L}(U)$ be the associated operad of linear isometric embeddings. Then it is a G-operad under the conjugation action, and it is always an N_∞ operad.

Definition 2.4. An H-set X of cardinality n is called *admissible* for $\mathcal{O}$ if the graph of the corresponding action homomorphism $H \to \Sigma_n$ is contained in $\mathcal{F}_n$.

Algebras R over an N_∞ operad $\mathcal{O}$ are G-equivariant E_∞ ring spectra which in addition admit coherent equivariant multiplications given by Hill-Hopkins-Ravenel norm maps [BH15b, Thm. 6.11]

$$N_K^H \colon \bigwedge_{H/K} \operatorname{Res}^G_K(R) \to \operatorname{Res}^G_H(R)$$

for those nested subgroups $K \leq H \leq G$ such that H/K is an admissible set for $\mathcal{O}$. (More generally, there is a norm map N_f associated to a map of G-sets $f \colon X \to Y$ provided that for all $y \in Y$, the preimage $f^{-1}(y)$ is an admissible G_y-set, where G_y denotes the stabilizer group of y.) Here, $\bigwedge_{H/K}$ denotes the *indexed smash product* or *Hill-Hopkins-Ravenel norm functor* [HHR16, Section 2.2.3], and the maps N_K^H arise as the counits of the adjunctions [HHR16, Prop. 2.27]

$$\bigwedge_{H/K}(-) \colon \mathbf{Comm^K} \rightleftarrows \mathbf{Comm^H} \colon \operatorname{Res}^H_K(-)$$

between categories of commutative monoids in equivariant spectra.

The data of admissible H-sets for all $H \leq G$ can be organized in the following way: For fixed H, the collection of admissible H-sets forms a symmetric monoidal subcategory of the category $\mathcal{S}et^H$ of finite H-sets under disjoint union. Together, these assemble into a subfunctor $\mathcal{I}$ of the coefficient system $\underline{\mathcal{S}et}$ whose value at G/H is the symmetric monoidal category $\mathcal{S}et^H$. The operad structure of $\mathcal{O}$ forces $\mathcal{I}$ to be closed under certain operations, as captured in the following definition.

Definition 2.5 ([BH16], Def. 1.2)**.** An *indexing system* is a contravariant functor

$$\mathcal{I} \colon \mathcal{O}rb_G^{op} \to \mathcal{S}ym, \; G/H \mapsto \underline{C}(H)$$

from the orbit category of G to the category of symmetric monoidal categories and strong symmetric monoidal functors, such that the following holds:

(i) The value $\mathcal{I}(H)$ of $\mathcal{I}$ at G/H is a full symmetric monoidal subcategory of the category $\mathcal{S}et^H$ of finite H-sets and H-equivariant maps which contains all trivial H-sets.
(ii) Each $\mathcal{I}(H)$ is closed under finite limits.
(iii) The functor $\mathcal{I}$ is closed under "self-induction": If $H/K \in \mathcal{I}(H)$ and $T \in \mathcal{I}(K)$, we require that $\mathrm{Ind}_K^H(T) = H \times_K T \in \mathcal{I}(H)$.

The collection of all indexing systems (for a fixed group G) forms a poset under inclusion. N_∞ operads give rise to indexing systems.

Definition 2.6. Let $\mathcal{I}$ be an indexing system. Call an H-set X *admissible* if $X \in \mathcal{I}(H)$. Call a map $f\colon Y \to Z$ of finite G-sets *admissible* if the orbit $G_{f(y)}/G_y$ obtained from stabilizer subgroups is admissible for all $y \in Y$.

Proposition 2.7 ([BH16], Thm. 4.17)**.** *The admissible sets of any N_∞ operad $\mathcal{O}$ form an indexing system.*

2.2. **The poset of N_∞ ring structures.** Two extreme cases of N_∞ operads arise:

Definition 2.8 ([BH15b], Section 3.1)**.** If for all $n \in \mathbb{N}$, $\mathcal{F}_n$ is the family of all graph subgroups of $G \times \Sigma_n$, then $\mathcal{O}$ is called a *G-E_∞ operad* or *complete N_∞ operad*. If for all n, $\mathcal{F}_n$ is the family of trivial graphs $H \times \{\mathrm{id}\}$, then $\mathcal{O}$ is called a *naive N_∞ operad*.

Algebras over G-E_∞ operads are equivariant E_∞ ring spectra which admit all norm maps and form a category which is Quillen equivalent to that of strict commutative monoids in G-spectra. Naive N_∞ operads are non-equivariant E_∞ operads equipped with the trivial G-action. Their algebras are all G-spectra that are underlying E_∞ ring spectra, but do not necessarily possess any non-trivial norms. The N_∞ operads with other collections of admissible sets interpolate between those two extremes. Up to homotopy, they form a poset that only depends on the combinatorial data of the admissible sets.

Definition 2.9 ([BH15b], Def. 3.9)**.** A morphism of N_∞ operads $\mathcal{O} \to \mathcal{O}'$ is a *weak equivalence* if it induces a weak equivalence of spaces $\mathcal{O}(n)^\Gamma \to \mathcal{O}'(n)^\Gamma$ for all $n \geq 0$ and all subgroups $\Gamma \leq G \times \Sigma_n$.

Blumberg-Hill conjectured the following equivalence of categories and proved the fully faithfulness [BH15b, Thm. 3.24]. Different proofs for the essential surjectivity were given by Gutierrez-White [GW17, Thm. 4.7], Rubin [Rub17, Thm. 3.3] and Bonventre-Pereira [BP17, Cor. IV], and it should be possible to extract an ∞-categorical proof from [BDG$^+$17] and its sequels.

Theorem 2.10 (Blumberg-Hill et al.)**.** *The functor from the homotopy category of N_∞ operads (with respect to the above notion of weak equivalence) to the poset of indexing systems which assigns to each N_∞ operad its collection of admissible sets is an equivalence of categories.*

Remark 2.11. We record a technical detail for later reference: [GW17, Thm. 4.10] guarantees that for each indexing system $\mathcal{I}$, we can find a corresponding N_∞ operad $\mathcal{O}$ which is Σ-*cofibrant*, i.e., each $\mathcal{O}(n)$ has the homotopy type of a (necessarily Σ_n-free) $(G \times \Sigma_n)$-CW complex. This will be used in Section 2.4.

2.3. **Mackey functors, Green rings and (incomplete) Tambara functors.** Recall that a *Mackey functor* $\underline{M}$ (with respect to an ambient group G which we leave implicit in the notation) consists of an abelian group $\underline{M}(T)$ for each finite G-set, equipped with a structure map $\underline{M}(X) \to \underline{M}(Z)$ for each span

$$X \xleftarrow{r} Y \xrightarrow{t} Z,$$

subject to a list of axioms. In particular, $\underline{M}$ is additive in the sense that $\underline{M}(S \sqcup T) \cong \underline{M}(S) \times \underline{M}(T)$. Thus, it is determined on objects by the values $\underline{M}(H) := \underline{M}(G/H)$ for subgroups $H \leq G$. We refer to [Str12, Section 3] for details.

A Mackey functor $\underline{R}$ is a *Green ring* if $\underline{R}(X)$ is a commutative ring for all G-sets X such that all restriction maps are ring homomorphisms and all transfers are homomorphisms of modules over the target.

Many naturally occuring examples of Green rings such as the Burnside ring $A(-)$ or the complex representation ring $RU(-)$ come equipped with additional multiplicative transfers, called *norms*. Green rings with compatible norms are known as *Tambara functors* (originally defined as "TNR functors" [Tam93]) and were generalized in [BH16] to cases where only some of the norm maps are available. We quickly review these *incomplete Tambara functors*.

Let $bispan^G$ denote the category of *bispans of G-sets*. It has objects the finite G-sets and morphisms the isomorphism classes of bispans of finite G-sets

$$X \xleftarrow{r} Y \xrightarrow{n} Z \xrightarrow{t} W.$$

We refer to [Str12, Section 6] for the definition of composition and further details. Blumberg-Hill observed that one can restrict the class of maps n which are allowed at the central position of a bispan to encode Tambara functors with incomplete collections of norms, as we recall now.

Definition 2.12 ([BH16], Sections 2.1, 3.2)**.** A subcategory D of $\mathcal{S}et^G$ is called

1) *wide* if it contains all objects,
2) *pullback-stable* if any base-change of a map in D is again in D, and
3) *finite coproduct-complete* if it has all finite coproducts and they are created in $\mathcal{S}et^G$.

Theorem 2.13 ([BH16], Thm. 2.10)**.** *Let $D \subseteq \mathcal{S}et^G$ be a wide, pullback-stable subcategory, then the wide subgraph $bispan_D^G$ of the category of bispans that only contains morphisms of the form*

$$X \leftarrow Y \to Z \to W$$

where $Y \to Z$ is in D, forms a subcategory.

Definition 2.14 ([BH16], Def. 3.8)**.** For an indexing system $\mathcal{I}$, let $\mathcal{S}et_{\mathcal{I}}^G \subseteq \mathcal{S}et^G$ be the wide subgraph which contains a morphism $f\colon X \to Y$ if and only if for all $y \in Y$, the quotient of stabilizers $G_{f(y)}/G_y$ is in $\mathcal{I}(G_{f(y)})$.

Theorem 2.15 ([BH16], Thm. 3.17)**.** *The assignment $\mathcal{I} \mapsto \mathcal{S}et_{\mathcal{I}}^G$ gives rise to an isomorphism between the poset of indexing systems and the poset of wide, pullback-stable, finite coproduct-complete subcategories $D \subseteq \mathcal{S}et^G$.*

Definition 2.16 ([BH16], Def. 4.1)**.** Let $D \subseteq \mathcal{S}et^G$ be a wide, pullback-stable symmetric monoidal subcategory.

1) A *D-semi Tambara functor* is a product-preserving functor $bispan_D^G \to \mathcal{S}et$.
2) A *D-Tambara functor* is a D-Tambara functor that is abelian group valued on objects.
3) For an indexing system $\mathcal{I}$ and $D = \mathcal{S}et_{\mathcal{I}}^G$, define an *$\mathcal{I}$-Tambara functor* to be a D-Tambara functor.
4) If $D = \mathcal{S}et^G$, then D-Tambara functors are simply called Tambara functors.

Remark 2.17. We did not require that D be finite coproduct-complete in the definition. If this also holds, i.e., if D corresponds to an indexing system $\mathcal{I}$, then it can be shown that every $\mathcal{I}$-Tambara functor has an underlying Green ring and all norm maps are maps of multiplicative monoids, see [BH16, Prop. 4.6 and Cor. 4.8].

The condition that any D-Tambara functor $\underline{R}$ be product-preserving means that

$$\underline{R}(S \sqcup T) \cong \underline{R}(S) \times \underline{R}(T)$$

for all finite G-sets S and T. Hence, on the level of objects, $\underline{R}$ is determined by the groups $\underline{R}(H) := \underline{R}(G/H)$ for all $H \leq G$.

Notation 2.18. We will use the following special cases of the structure maps frequently in the present paper: Spans of the form $(Y \xleftarrow{f} X \xrightarrow{\mathrm{id}} X \xrightarrow{\mathrm{id}} X)$ give rise to *restrictions* $R_f\colon \underline{R}(Y) \to \underline{R}(X)$ and spans of the form $(X \xleftarrow{\mathrm{id}} X \xrightarrow{\mathrm{id}} X \xrightarrow{f} Y)$ induce *transfers*

$T_f\colon \underline{R}(X) \to \underline{R}(Y)$. Moreover, spans of the form $(X \xleftarrow{\mathrm{id}} X \xrightarrow{f} Y \xrightarrow{\mathrm{id}} Y)$ give rise to *norms* $N_f\colon \underline{R}(X) \to \underline{R}(Y)$. If $f\colon X \to Y$ is the canonical surjection $G/K \to G/H$ arising from nested subgroup inclusions $K \le H \le G$, then we write $R_K^H := R_f$, $T_K^H := T_f$ and $N_K^H := N_f$, respectively.

Example 2.19. The Burnside ring $A(G)$ is a Tambara functor. Restrictions, transfers and norms are given by restriction, induction and co-induction of G-sets, respectively. Similarly, the complex representation ring $RU(G)$ is a Tambara functor with restrictions, transfers and norms given by restriction, induction and tensor induction of G-representations, respectively. The "linearization map" $A(G) \to RU(G)$ that sends a finite G-set to its associated permutation representation is a map of Tambara functors, i.e., it is compatible with all of the structure maps.

Another class of examples arises from equivariant stable homotopy theory: The norm maps N_K^H of an N_∞ ring spectrum R give rise to multiplicative transfers on equivariant homotopy groups

$$N_K^H\colon \pi_V^K(R) \to \pi^H_{\mathrm{Ind}_K^H(V)}(\bigwedge_{H/K} R)$$

given by sending the K-equivariant homotopy class of $f\colon \mathbb{S}^V \to R$ to the H-equivariant homotopy class of the composite

$$\mathbb{S}^{\mathrm{Ind}_K^H(V)} \cong \bigwedge_{H/K} \mathbb{S}^V \xrightarrow{\wedge f} \bigwedge_{H/K} R \xrightarrow{N_K^H} R,$$

see [HHR16, Section 2.3.3].

Theorem 2.20 ([Bru07], [BH16], Thm. 4.14)**.** *Let R be an algebra over an N_∞ operad $\mathcal{O}$, then $\underline{\pi}_0(R)$ is an $\mathcal{I}$-Tambara functor structured by the indexing system $\mathcal{I}$ corresponding to $\mathcal{O}$ under the equivalence of categories from Theorem 2.10.*

The structure on the entire homotopy ring $\{\pi_V^H(R)\}_{H\le G,\, V\in RO(H)}$ is described in [AB15]. For the purpose of the present paper, it suffices to consider the zeroth equivariant homotopy groups.

2.4. **Localization and N_∞ rings.** We record some preservation results for algebraic structure under localizations of G-spectra which invert a single element $x \in \pi_*^G(\mathbb{S})$. For definiteness, we work in the category of orthogonal G-spectra equipped with the positive complete model structure [HHR16, Thm. B.63].

Notation 2.21. By abuse of notation, let x be a map representing the homotopy class $x \in \pi_0^G(\mathbb{S})$. We write $\mathcal{C}_x$ for the set of morphisms of G-spectra

$$\mathcal{C}_x = \{G_+ \wedge_H S^n \wedge x \mid H \le G,\, n \in \mathbb{Z}\}.$$

Proposition 2.22. *Bousfield localization at $\mathcal{C}_x$ has the following properties:*

i) It is given by smashing with $\mathbb{S}[x^{-1}]$, hence recovers (orbitwise) x-localization on the level of equivariant homotopy groups.
ii) It is a monoidal *localization in the sense that the resulting local model structure is again a monoidal model category.*

Proof. Since the map $ho(\mathrm{Sp}^G)(G/H_+ \wedge \mathbb{S}^n \wedge x, X)$ is just the action of x on $\pi_n^H(X)$, we see that an object X is $\mathcal{C}_x$-local if and only if its equivariant homotopy groups are x-local. But x-localization is given by smashing with $\mathbb{S}[x^{-1}]$. By [Whi14, Thm. 4.5], the localization is monoidal if and only if $\mathcal{C}_x$ is closed under all functors $G_+ \wedge_H S^n \wedge (-)$, which holds by definition. □

Even such a seemingly innocent localization need not preserve any of the (non-trivial) norm maps of an N_∞ ring spectrum, as the following example illustrates.

Example 2.23 ([HH16], Prop. 6.1)**.** The inclusion of 0 into the reduced regular representation $\tilde{\rho}$ of G defines an essential map $S^0 \to S^{\tilde{\rho}}$ of G-spaces all of whose restrictions to proper subgroups are equivariantly contractible because they necessarily have fixed points along which the two points of S^0 can be connected by an equivariant path. The resulting map gives rise to an element $\alpha \in \pi^G_{-\tilde{\rho}}(\mathbb{S})$ such that the resulting G-spectrum $\mathbb{S}[\alpha^{-1}]$ is non-trivial but all of its restrictions to proper subgroups are equivariantly contractible. Thus, it cannot admit any norms

$$\bigwedge_{G/H} \mathrm{Res}^G_H \mathbb{S}[\alpha^{-1}] \to \mathbb{S}[\alpha^{-1}]$$

because on homotopy rings, they would induce ring maps from zero rings to non-trivial rings.

Remark 2.24. Strictly speaking, the element α is not an element of the $\mathbb{Z}$-graded homotopy groups $\pi_*^G(\mathbb{S})$, but only of the $RO(G)$-graded homotopy groups $\pi_\star^G(\mathbb{S})$. However, our results in Section 4 show that even when we restrict attention to elements $x \in \pi_0^G(\mathbb{S})$, we can construct many other examples of the loss of N_∞ structure under localization in terms of elementary group theory. Indeed, the A_5-spectra $\mathbb{S}[e_{A_5}^{-1}]$ and $\mathbb{S}[\alpha^{-1}]$ are very similar in terms of their equivariant multiplicative behavior, see Section 5.

We now present a preservation result for N_∞ ring structures due to Gutierrez and White. A similar result first appeared in the special case of G-E_∞ rings in [HH14, Cor. 4.11] for G-E_∞ ring spectra and goes back at least to [EKMM97, Thm. VIII.2.2].

Definition 2.25 ([GW17], Def. 7.3)**.** For a G-operad $\mathcal{P}$, let U denote the forgetful functor from $\mathcal{P}$-algebras to G-spectra. A Bousfield localization $L_{\mathcal{C}}$ is said to *preserve* $\mathcal{P}$-algebras if the following two conditions hold:

(1) If E is a $\mathcal{P}$-algebra, then there is some $\mathcal{P}$-algebra $\tilde{E}$ which is weakly equivalent as a G-spectrum to $L_{\mathcal{C}}(E)$.
(2) In addition, if E is a cofibrant $\mathcal{P}$-algebra, then there is a choice of $\tilde{E}$ in the category of $\mathcal{P}$-algebras with $U(\tilde{E})$ local in G-spectra, there is a $\mathcal{P}$-algebra homomorphism $r_E\colon E \to \tilde{E}$ that lifts the localization map $l_E\colon E \to L_{\mathcal{C}}(E)$ up to homotopy, and there is a weak equivalence $\beta_E\colon L_{\mathcal{C}}(UE) \to U(\tilde{E})$ such that $\beta_E \circ l_{UE} \cong Ur_E$ in $ho(\mathrm{Sp}^G)$.

Recall that a G-operad $\mathcal{P}$ is called Σ-*cofibrant* if all of its spaces $\mathcal{P}(n)$ have the homotopy type of $(G \times \Sigma_n)$-CW complexes. The following preservation result is a direct translation of [GW17, Cor. 7.10] to the positive complete model structure on orthogonal G-spectra.

Theorem 2.26 ([GW17], Cor. 7.10)**.** *Let $\mathcal{P}$ be a Σ-cofibrant N_∞ operad. Let $L_{\mathcal{C}}$ be a monoidal left Bousfield localization. Then $L_{\mathcal{C}}$ preserves $\mathcal{P}$-algebras in G-spectra if and only if the functors*

$$G_+ \wedge_H \bigwedge_{H/K} \mathrm{Res}^G_K(-)\colon \mathrm{Sp}^G \to \mathrm{Sp}^G$$

preserve $\mathcal{C}$-local equivalences between cofibrant objects for all $H \leq G$ and all transitive H-sets H/K which are admissible for $\mathcal{P}$.

Remark 2.27. Note that our formulation of Theorem 2.26 fixes a minor error in the statement of [GW17, Cor. 7.10]: It is clear from their proof that one has to take into account all norms given by admissible H-sets H/K for all $H \leq G$ (and not just norms from G-sets G/H), and that one only obtains a statement about the local equivalences between cofibrant objects.

Corollary 2.28 ([GW17], Cor. 7.5)**.** *Let $\mathcal{P}$ be any N_∞ operad. Then any monoidal left Bousfield localization $L_{\mathcal{C}}$ takes $\mathcal{P}$-algebras in Sp^G to G-spectra which are at least algebras over some naive N_∞ operad.*

If the localization is given by inverting a single element $x \in \pi_0^G(\mathbb{S})$, the condition in Theorem 2.26 can be verified on homotopy groups, as we explain now. The following results generalize [HH14, Thm. 4.11] to the N_∞ setting in the case of the (p-local) sphere spectrum.

Lemma 2.29. *Let R be a cofibrant commutative monoid in Sp^G. Fix a transitive H-set H/K and an element $x \in \pi_0^G(R)$. Then there is an equivalence of H-spectra*

$$\bigwedge_{H/K} \mathrm{Res}^G_K(R[x^{-1}]) \simeq \Big(\bigwedge_{H/K} \mathrm{Res}^G_K(R)\Big)\Big[\Big(\bigwedge_{H/K} \mathrm{Res}^G_K(x)\Big)^{-1}\Big].$$

Here, the localization $R[x^{-1}]$ is interpreted as the sequential homotopy colimit along countably many copies of the "multiplication by x" map

$$\cdot x\colon R \wedge \mathbb{S} \xrightarrow{R\wedge x} R \wedge R \xrightarrow{mult} R.$$

The localization on the right hand side is obtained similarly, replacing $\cdot x$ by $\bigwedge_{H/K}(\cdot x)$.

Proof. Both the restriction functor and the norm functor admit left derived functors which commute with sifted homotopy colimits. This is well-known for the restriction; for the norm it follows from [HHR16, Prop. B.105] combined with [HHR16, Prop. A.27, A.53]. Thus, the functor $\bigwedge_{H/K} \mathrm{Res}^G_K$ admits a left derived functor $\mathbb{L}(\bigwedge_{H/K} \mathrm{Res}^G_K)$ which preserves the sequential homotopy colimits that compute localizations. It follows that $\mathbb{L}(\bigwedge_{H/K} \mathrm{Res}^G_K)(R[x^{-1}])$ is the sequential homotopy colimit along copies of the map

$$\mathbb{L}(\bigwedge_{H/K} \mathrm{Res}^G_K)(\cdot x)\colon \mathbb{L}(\bigwedge_{H/K} \mathrm{Res}^G_K)(R) \to \mathbb{L}(\bigwedge_{H/K} \mathrm{Res}^G_K)(R)$$

Let $R_c \to R$ be a cofibrant approximation of the underlying G-spectrum of R, then we may take $R_c[x^{-1}] \to R[x^{-1}]$ to be a cofibrant approximation of its x-localization. Now we make use of our cofibrancy assumption and apply [HHR16, Prop. 2.30] twice: Applied to the map $R_c \to R$, we see that

$$\mathbb{L}(\bigwedge_{H/K} \mathrm{Res}^G_K)(R) \simeq \bigwedge_{H/K} \mathrm{Res}^G_K(R_c) \simeq \bigwedge_{H/K} \mathrm{Res}^G_K(R)$$

and hence the above homotopy colimit is seen to be $(\bigwedge_{H/K} \mathrm{Res}^G_K(R))[(\bigwedge_{H/K} \mathrm{Res}^G_K(x))^{-1}]$. Applied to $R_c[x^{-1}] \to R[x^{-1}]$, we obtain equivalences

$$\mathbb{L}(\bigwedge_{H/K} \mathrm{Res}^G_K)(R[x^{-1}]) \simeq \bigwedge_{H/K} \mathrm{Res}^G_K(R_c[x^{-1}]) \simeq \bigwedge_{H/K} \mathrm{Res}^G_K(R[x^{-1}]),$$

which completes the proof. □

Proposition 2.30. *Let $\mathcal{P}$ be a Σ-cofibrant N_∞ operad. Fix $x \in \pi_0^G(\mathbb{S})$. Then $L_{\mathcal{C}_x}$ preserves $\mathcal{P}$-algebras in G-spectra if and only for all $H \leq G$ and all transitive admissible H-sets H/K, the element $N_K^H R_K^G(x)$ divides a power of $R_H^G(x)$ in the ring $\pi_0^H(\mathbb{S})$.*

Proof. We have to show that for admissible such H/K, the functors $G_+ \wedge_H \bigwedge_{H/K} \mathrm{Res}^G_K(-)$ preserve $\mathcal{C}_x$-local equivalences between cofibrant objects if and only if the elements $N_K^H R_K^G(x)$ divide powers of $R_H^G(x)$.

If $\mathcal{C}_x$-local equivalences are preserved, then in particular the map of G-spectra

$$G_+ \wedge_H \bigwedge_{H/K} \mathrm{Res}^G_K(x)\colon G_+ \wedge_H \bigwedge_{H/K} \mathrm{Res}^G_K(\mathbb{S}) \to G_+ \wedge_H \bigwedge_{H/K} \mathrm{Res}^G_K(\mathbb{S})$$

is an x-local equivalence. Under the standard isomorphism $\pi_*^G(G_+ \wedge_H -) \cong \pi_*^H(-)$, the induced map on $\pi_*^G(-)$ agrees with multiplication by the element $N_K^H R_K^G(x)$ and becomes a unit after inverting $R_H^G(x)$, hence the element $N_K^H R_K^G(x)$ must divide a power of $R_H^G(x)$.

Conversely, assume the division relation holds and let $f\colon X \to Y$ be a $\mathcal{C}_x$-local equivalence between cofibrant objects. Since induction is a left Quillen functor, it suffices to show that the map $\bigwedge_{H/K} \operatorname{Res}_K^G(f)$ becomes an equivalence of H-spectra upon smashing with $\mathbb{S}[R_H^G(x)^{-1}]$. We are going to show that it is an equivalence upon smashing with (a cofibrant replacement of) $\mathbb{S}[N_K^H R_K^G(x)^{-1}]$. Since the element $N_K^H R_K^G(x)$ divides $R_H^G(x)$ by assumption, the claim then follows.

Choose a cofibrant replacement $\mathbb{S}_c \to \mathbb{S}$. The map $f \wedge \mathbb{S}_c[x^{-1}]$ is an equivalence of cofibrant G-spectra, so $\bigwedge_{H/K} \operatorname{Res}_K^G(f \wedge \mathbb{S}_c[x^{-1}])$ is an equivalence of H-spectra by [HHR16, Prop. B.104]. Finally, we have equivalences of maps of H-spectra

$$\bigwedge_{H/K} \operatorname{Res}_K^G(f \wedge \mathbb{S}_c[x^{-1}]) \cong \left(\bigwedge_{H/K} \operatorname{Res}_K^G(f)\right) \wedge \left(\bigwedge_{H/K} \operatorname{Res}_K^G(\mathbb{S}_c[x^{-1}])\right)$$
$$\simeq \left(\bigwedge_{H/K} \operatorname{Res}_K^G(f)\right) \wedge \mathbb{S}_c[(N_K^H R_K^G(x))^{-1}]$$

where the latter is given by Lemma 2.29. □

Corollary 2.31. *Let $n \in \mathbb{Z}$, viewed as the element $n \cdot [G/G] \in A(G)$. Then $\mathbb{S}[\frac{1}{n}]$ is a complete G-E_∞ ring spectrum.*

Proof. This is claimed without proof at the very end of [HH14] and proven in [BH17, Lemma 12.8]. Alternatively, one can just verify the conditions of Proposition 2.32 using e.g. the formula given in [Oda14, Lemma 2.3]. □

Consequently, for any collection $\mathfrak{p}$ of primes, $\mathbb{S}_{(\mathfrak{p})} := \mathbb{S}[q^{-1}, q \notin \mathfrak{p}]$ is a complete G-E_∞ ring spectrum, or equivalently, a commutative monoid in Sp^G. One can now mimick the proof of Proposition 2.30 in the $\mathfrak{p}$-local case.

Proposition 2.32. *Let $\mathcal{P}$ be a Σ-cofibrant N_∞ operad. Fix $x \in \pi_0^G(\mathbb{S}_{(\mathfrak{p})})$. Then $L_{\mathcal{C}_x}$ preserves $\mathcal{P}$-algebras in $\mathfrak{p}$-local G-spectra if and only for all $H \leq G$ and all admissible H-sets T, the element $N_K^H R_K^G(x)$ divides a power of $R_H^G(x)$ in the ring $\pi_0^H(\mathbb{S}_{(\mathfrak{p})})$.*

2.5. **Localization and incomplete Tambara functors.** There are analogous preservation results for incomplete Tambara functors under localization. Given an $\mathcal{I}$-Tambara functor $\underline{R}$ and an element $x \in \underline{R}(G)$, consider the levelwise localization $\underline{R}[x^{-1}](H) := \underline{R}(H)[R_H^G(x)^{-1}]$. By [Str12, Lemma 10.2], this agrees with the sequential colimit along countably many copies of multiplication by x, taken in the category of Mackey functors. Multiplication by x is typically not a map of Tambara functors, and the levelwise localization is usually not a Tambara functor. An alternative notion of localization which enjoys a universal property in the category of Tambara functors is discussed in [BH16, Section 5.4]. The two notions agree if and only if the Hill-Hopkins conditions are satisfied.

Theorem 2.33 ([BH16], Thm. 5.25)**.** *Let $\underline{R}$ be an $\mathcal{I}$-Tambara functor structured by an indexing system $\mathcal{I}$. Let $x \in \underline{R}(G)$. Then the orbit-wise localization $\underline{R}[x^{-1}]$ is a localization in the category of $\mathcal{I}$-Tambara functors if and only if for all admissible sets H/K of $\mathcal{I}$, the element $N_K^H R_K^G(x)$ divides a power of $R_H^G(x)$.*

Blumberg and Hill do not give a detailed proof in [BH16], but assert that the proof strategy of [HH14] can be mimicked in the setting of incomplete Tambara functors. For completeness, we include a (different and more elementary) proof here.

Proof. As before, we decorate the structure maps of the localization with a tilde. In order to simplify notation, write $x_K := R_K^G(x)$ and similar for H. Fix an admissible set $H/K \in \mathcal{I}(H)$. If $\tilde{N} := \tilde{N}_K^H$ exists, it must necessarily be given as

$$\tilde{N}\left(\frac{a}{x_K^n}\right) = \frac{N(a)}{N(x_K)^n}. \tag{2.34}$$

This expression is well-defined if and only $N(x_K) \in \underline{R}(H)$ becomes a unit after inverting x_H, i.e., if and only if it divides a power of x_H.

Thus, $\underline{R}[x^{-1}]$ is a Green ring equipped with norms $\tilde{N}_K^H$ for all admissible sets $H/K \in \mathcal{I}(H)$ and all $H \leq G$. From (2.34) we see that the reciprocity relations [BH16, Prop. 4.10, Prop. 4.11] satisfied by the norms of $\underline{R}$ imply the reciprocity relations for the norms of $\underline{R}[x^{-1}]$. Thus by [BH16, Thm. 4.13], $\underline{R}[x^{-1}]$ is a $\mathcal{I}$-Tambara functor. Moreover, the canonical map $\underline{R} \to \underline{R}[x^{-1}]$ is a map of $\mathcal{I}$-Tambara functors. One readily verifies that the unique ring maps out of $\underline{R}(H)[x_H^{-1}]$ given by the universal properties for varying $H \leq G$ assemble into a map of $\mathcal{I}$-Tambara functors which exhibits $\underline{R}[x^{-1}]$ as the localization of $\underline{R}$ at x.

For the "only if" direction, observe that the division relations are also necessary because the norms and restrictions of the incomplete Tambara functor $\underline{R}[x^{-1}]$ are multiplicative maps. □

As before, this always applies to localizations which invert natural numbers.

Corollary 2.35. *Let $n \in \mathbb{Z}$, viewed as the element $n \cdot [G/G] \in A(G)$. Then $A(-)[\frac{1}{n}]$ is a complete Tambara functor.*

In particular, Question 1.3 also makes sense for the local variants of the Burnside ring.

3. Idempotent splittings of the Burnside ring and the G-sphere spectrum

We review Dress' classification of idempotents in the ($\mathfrak{p}$-local) Burnside ring and describe the resulting product decompositions of the Burnside Mackey functor and the G-equivariant sphere spectrum. All of the statements in this section are easy consequences of Dress' result and are probably well-known to the experts. The author does not claim any originality for these results.

3.1. **Idempotents in the Burnside ring.** Let $\mathfrak{p}$ be a collection of prime numbers and set $\mathbb{Z}_{(\mathfrak{p})} := \mathbb{Z}\left[p^{-1} \mid p \notin \mathfrak{p}\right]$. If $\mathfrak{p}$ is the collection of all primes, nothing is inverted and hence $\mathbb{Z}_{(\mathfrak{p})} = \mathbb{Z}$. If $\mathfrak{p}$ is the empty set, then all primes are inverted, hence $\mathbb{Z}_{(\mathfrak{p})} = \mathbb{Q}$. For $\mathfrak{p} = \{p\}$, we obtain the usual p-localization $\mathbb{Z}_{(\mathfrak{p})} = \mathbb{Z}_{(p)}$, which justifies the notation. Write $A(G)_{(\mathfrak{p})} := A(G) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\mathfrak{p})}$ for the $\mathfrak{p}$-local Burnside ring.

Lemma 3.1 ([tD78], Prop. 1)**.** *Every finite group G has a unique minimal normal subgroup $O^{\mathfrak{p}}(G)$ such that the quotient $G/O^{\mathfrak{p}}(G)$ is a solvable $\mathfrak{p}$-group, i.e. a solvable group whose order is only divisible by primes in $\mathfrak{p}$.*

Definition 3.2. The group $O^{\mathfrak{p}}(G) \leq G$ is called the $\mathfrak{p}$-*residual subgroup* of G. A group G is called $\mathfrak{p}$-*perfect* if $G = O^{\mathfrak{p}}(G)$. If $\mathfrak{p}$ contains all primes, we will write $O^{\mathrm{solv}}(G) := O^{\mathfrak{p}}(G)$ for the minimal normal subgroup with solvable quotient.

Remark 3.3. The following statements are easily verified.

i) For $\mathfrak{p} = \{\text{all primes}\}$, this agrees with the usual definition of a perfect group.
ii) For $\mathfrak{p} = \{p\}$, the group $O^{\mathfrak{p}}(G)$ is known to group theorists as the p-residual subgroup $O^{p}(G)$ and the condition that the quotient be solvable is redundant since every finite p-group is solvable.
iii) For $\mathfrak{p} = \emptyset$, every finite group G is $\mathfrak{p}$-perfect because the trivial group is the only $\mathfrak{p}$-group.

The following classification result is due to Dress. Recall that the assignment $S \mapsto |S^H|$ given by taking the cardinality of the H-fixed points of a finite G-set S extends to an injective ring homomorphism

$$\phi^H \colon A(G) \to \prod_{(H) \leq G} \mathbb{Z}$$

where the product is taken over conjugacy classes of subgroups $H \leq G$ [Dre69, (4), (5), Lemma 1]. The same is true after inverting primes since $\mathbb{Z}_{(\mathfrak{p})}$ has no torsion. The number $\phi^H(x)$ is called the *mark* of x at H.

Theorem 3.4 ([Dre69], Prop. 2). *There is a bijection between the conjugacy classes of $\mathfrak{p}$-perfect subgroups $L \leq G$ and the set of primitive idempotent elements of $A(G)_{(\mathfrak{p})}$ which sends L to the element $e_L \in A(G)_{(\mathfrak{p})}$ whose marks $\phi^H(e_L)$ at a subgroup $H \leq G$ are one if $O^{\mathfrak{p}}(H) \sim L$ are conjugate in G, and zero otherwise.*

Remark 3.5. It follows immediately that G is solvable if and only if $A(G)$ does not have any non-trivial idempotents. This originally motivated Dress' work in [Dre69].

Remark 3.6. Note that if p does not divide the order G, then all subgroups $L \leq G$ are p-perfect, hence all idempotents of $A(G) \otimes \mathbb{Q}$ are contained in the subring $A(G)_{(p)}$. For the other extreme case, if G is a p-group, then only the trivial subgroup is p-perfect, hence the only idempotents in $A(G)_{(p)}$ are zero and one.

3.2. Idempotent splittings of the Burnside ring.

Corollary 3.7. *There is a bijection of rings*

$$A(G)_{(\mathfrak{p})} \cong \prod_{(L) \leq G} A(G)_{(\mathfrak{p})}[e_L^{-1}]$$

where the product is taken over conjugacy classes of perfect subgroups of $L \leq G$.

One readily verifies that the statement above can be upgraded to a splitting of Green rings, where for any subgroup $H \leq G$, we view e_L as an element of $A(G)_{(\mathfrak{p})}$ via the restriction map $R^G_H \colon A(G)_{(\mathfrak{p})} \to A(H)_{(\mathfrak{p})}$.

Notation 3.8. For brevity, we will write $A(-)_{(\mathfrak{p})}[e_L^{-1}]$ for the levelwise localization $A(-)_{(\mathfrak{p})}[R^G_{(-)}(e_L)^{-1}]$, see Section 2.5.

Proposition 3.9. *There is a bijection of Green rings*

$$A(-)_{(\mathfrak{p})} \cong \prod_{(L) \leq G} A(-)_{(\mathfrak{p})}[e_L^{-1}].$$

The left hand side is even a Tambara functor. Question 1.3 asks whether the factors on the right hand side inherit norms from $A(-)_{(\mathfrak{p})}$, and whether the splitting preserves these norms.

Remark 3.10. The value of the Green ring $A(-)_{(\mathfrak{p})}[e_L^{-1}]$ at a subgroup $K \leq G$ is non-zero if and only if L is subconjugate to K, as follows from the description of e_L in terms of marks in Theorem 3.4.

Remark 3.11. Note that for any idempotent $e \in A(G)_{(\mathfrak{p})}$, the localization $A(G)_{(\mathfrak{p})}[e^{-1}]$ is canonically isomorphic to the submodule $e \cdot A(G)_{(\mathfrak{p})}$. The restriction maps $\tilde{R}^H_K$ and transfer maps $\tilde{T}^H_K$ of $A(-)_{(\mathfrak{p})}[e^{-1}]$ are given by the formulae

$$\tilde{R}^H_K(R^G_H(e) \cdot a) := R^G_K(e) \cdot R^H_K(a)$$

and

$$\tilde{T}^H_K(R^G_K(e) \cdot b) := R^G_H(e) \cdot T^H_K(b)$$

for all $a \in A(H)$ and $b \in A(K)$, where R and T denote the restrictions and transfers of $A(-)_{(\mathfrak{p})}$ (cf. [LMS86, Thm. V.4.6]). The equations that go into verifying the proposition can easily be read off from these formulae. The analogous $\mathfrak{p}$-local statements hold.

Remark 3.12. We warn the reader that even though any restriction of e_L to a proper subgroup $H \leq G$ is still an idempotent, it will in general not be primitive. More precisely, it splits as an n-fold sum of primitive idempotents of $A(H)_{(\mathfrak{p})}$ where n is the number of H-conjugacy classes contained in the G-conjugacy class of L.

3.3. **Idempotent splittings of the sphere spectrum.** We now turn to the homotopical consequences of the above splitting. First recall the following theorem which goes back to Segal [Seg71, Cor. of Prop. 1].

Theorem 3.13 (See [Sch], Thm. 6.14, Ex. 10.11)**.** *For all $H \leq G$, there is a ring isomorphism $A(H) \to \pi^H_0(\mathbb{S})$ which sends the class represented by H/K to the element $T^H_K(\mathrm{id})$. For varying H, these maps assemble into an isomorphism of Tambara functors $\underline{\pi}_0(\mathbb{S}) \cong A(-)$.*

Remark 3.14. The isomorphism $A(-) \cong \underline{\pi}_0(\mathbb{S})$ is completely determined by the requirements that it be unital and respect transfers.

Dress' classification of idempotent elements then immediately implies the next statement.

Proposition 3.15. *The product of the canonical maps to the localizations is a weak equivalence of $\mathfrak{p}$-local G-spectra*

$$\mathbb{S}_{(\mathfrak{p})} \simeq \prod_{(L) \leq G} \mathbb{S}_{(\mathfrak{p})}[e_L^{-1}]$$

where the product is taken over conjugacy classes of $\mathfrak{p}$-perfect subgroups. For any naive N_∞-operad $\mathcal{O}$, i.e., any N_∞ operad whose homotopy type is the unique minimal element in the poset of N_∞ ring structures, this is a splitting of $\mathcal{O}$-algebras (up to equivalence of G-spectra).

Proof. The fact that the e_L form a complete set of orthogonal idempotents implies that the map induces isomorphisms on all equivariant homotopy groups. Moreover, for any naive N_∞ operad $\mathcal{O}$, the G-homotopy equivalence $\mathcal{O}(0) \to *$ induces a canonical equivalence of G-spectra $(\Sigma^\infty_+\mathcal{O}(0))_{(\mathfrak{p})} \to \mathbb{S}_{(\mathfrak{p})}$, so we can view the latter as an $\mathcal{O}$-algebra. Under this identification, the canonical maps $\mathbb{S}_{(\mathfrak{p})} \to \mathbb{S}_{(\mathfrak{p})}[e_L^{-1}]$ are all maps of $\mathcal{O}$-algebras, as follows from the fact that localization always preserves naive N_∞ rings, see Corollary 2.28. $\square$

Question 1.2 asks about the maximal N_∞ ring structures on the localizations $\mathbb{S}_{(\mathfrak{p})}[e_L^{-1}]$, and about the maximal N_∞ ring structure preserved by the splitting. The answer is given in Corollary 4.26 (Corollary F) and Corollary 4.29 (Corollary H).

4. Norms in the idempotent splittings

We state and prove the results which answer Question 1.3 and Question 1.2, including the local variants where any collection of primes $\mathfrak{p}$ is inverted.

4.1. Theorem A and consequences.

The main combinatorial result of this paper is the following version of Theorem A, stated in full $\mathfrak{p}$-local generality:

Theorem 4.1. *Let $\mathfrak{p}$ be a collection of primes. Let $L \leq G$ be a $\mathfrak{p}$-perfect subgroup and let $e_L \in A(G)_{(\mathfrak{p})}$ be the corresponding primitive idempotent under the bijection from Theorem 3.4. Fix subgroups $K \leq H \leq G$. Then the norm map $N_K^H\colon A(K)_{(\mathfrak{p})} \to A(H)_{(\mathfrak{p})}$ descends to a well-defined map of multiplicative monoids*

$$\tilde{N}_K^H\colon A(K)_{(\mathfrak{p})}[e^{-1}] \to A(H)_{(\mathfrak{p})}[e^{-1}]$$

if and only if one of the following holds:

i) Neither K nor H are super-conjugate in G to L.
ii) Both K and H are super-conjugate in G to L and satisfy the following condition:
($\star$) If $L' \leq G$ is conjugate in G to L and is contained in H, then it is contained in K.

The characterization of e_L in terms of marks in Theorem 3.4 implies that $R_H^G(e_L) = 0$ whenever L is not subconjugate in G to H. From this, it is clear that in case i) of Theorem 4.1, the norm $\tilde{N}_K^H$ exists for trivial reasons: it is just the zero morphism between zero rings. Similarly, there cannot be a norm map $\tilde{N}_K^H$ inherited from N_K^H if K is not super-conjugate to L, but H is. Indeed, it would have to be a map of multiplicative monoids from the zero ring to a non-trivial ring, hence would satisfy $\tilde{N}_K^H(0) = 1$. But $N_K^H(0) = [\mathrm{map_K}(\mathrm{H}, \emptyset)] = 0$ before localizing, which is a contradiction. The condition in case ii) is not obvious. We defer the proof of Theorem 4.1 to Section 4.2 and first state and prove the locally enhanced versions of Corollary B and Corollary C.

Corollary 4.2. *Assume that $L \leq G$ is normal and $\mathfrak{p}$-perfect. Fix $K \leq H \leq G$. Then $A(-)_{(\mathfrak{p})}[e_L^{-1}]$ inherits a norm $\tilde{N}_K^H$ from $A(-)_{(\mathfrak{p})}$ if and only if one of the following two conditions is satisfied:*

i) Neither K nor H contain any G-conjugate of L as a subgroup.
ii) Both K and H contain some G-conjugate of L as a subgroup.

Conversely, if for some $\mathfrak{p}$-perfect $L \leq G$ the two conditions are equivalent to the existence of $\tilde{N}_K^H$, then L is necessarily normal in G.

Proof. Let $L \leq K \leq H$. Then L is the only group in its G-conjugacy class, hence $(\star)$ is always satisfied for such K and H. Conversely, if $(\star)$ holds for $K := L$ and $H := G$, then any G-conjugate of L is contained in L, hence L is normal in G. ☐

Corollary 4.3. *The Green ring $A(-)_{(\mathfrak{p})}[e_L^{-1}]$ admits norms $\tilde{N}_K^H$ for all $K \leq H$ if and only if $L = 1$ is the trivial group. In this case, the norm maps equip $A(-)_{(\mathfrak{p})}[e_1^{-1}]$ with the structure of a Tambara functor.*

Proof. If $L = 1$, then all groups are supergroups of L and all subgroup inclusions give rise to norm maps by Theorem 4.2. It then follows from [BH16, Thm. 4.13] that $A(-)_{(\mathfrak{p})}[e_1^{-1}]$ is a Tambara functor, cf. the proof of Theorem 2.33. Conversely, if L is non-trivial $\mathfrak{p}$-perfect, the inclusion $1 \to G$ does not give rise to a well-defined norm on $A(-)_{(\mathfrak{p})}[e_L^{-1}]$. ☐

Remark 4.4. The "only if" part is implicit in work of Blumberg and Hill, at least integrally: All idempotents e_L different from e_1 lie in the augmentation ideal of $A(G)$. If inverting such an element yielded a Tambara functor, then it would have to be the zero Tambara functor, see [BH16, Example 5.24]. But $A(-)[e_L^{-1}]$ is always non-zero.

Remark 4.5. It is also implicit in Nakaoka's work on ideals of Tambara functors [Nak12] that the idempotent summands of the ($\mathfrak{p}$-local) Burnside ring Mackey functor cannot all be Tambara functors, for if they were, then the idempotent splitting would be a splitting of Tambara functors. But by [Nak12, Prop. 4.15], this implies that $A(1) \cong \mathbb{Z}$ splits non-trivially, which is absurd. (Note that there is a minor error in statements (2)–(4) of loc. cit.: the requirement that the respective ideals and elements be non-zero is missing.)

Remark 4.6. When working p-locally, the ring $A(G)_{(p)}[e_1^{-1}]$ can be described in two different ways: It agrees with the p-local Burnside ring with p-isotropy, i.e., the p-localization of the Grothendieck ring of finite G-sets all of whose isotropy groups are p-groups. Moreover, it can be identified with the p-localization of the Burnside ring of the p-fusion system of the group G. We refer the reader to [Gro, Section 5] for details.

As an illustration of Theorem 4.1, we will discuss the idempotent splittings of $A(A_5)$ (integrally) and $A(\Sigma_3)$ (locally at the primes 2 and 3) in detail in Section 5. There, we also spell out what happens in the rational splitting ($\mathfrak{p} = \emptyset$) for any finite group G.

4.2. **The proof of Theorem A.** The main idea of the proof is that we can check the hypotheses for preservation of norm maps from Theorem 2.33 on marks. As norm maps in the Burnside ring are given by co-induction functors of equivariant sets, we need to understand how they interact with taking fixed points. To that end, we first record some technical statements before giving the proof of Theorem 4.1 (Theorem A).

Lemma 4.7. *For subgroups $K, H \leq N \leq G$, let P be the pullback in the category of G-sets of the canonical surjections $G/H \to G/N$ and $G/K \to G/N$.*

$$\begin{array}{ccc} P & \longrightarrow & G/H \\ \downarrow & & \downarrow \\ G/K & \longrightarrow & G/N \end{array}$$

Then P has an orbit decomposition given by

$$P \cong \coprod_{n \in K \backslash N / H} G/(K \cap {}^{n}H)$$

where the summation is over representatives of double cosets.

This implies a multiplicative double coset formula for norm maps of $A(-)_{(\mathfrak{p})}$.

Corollary 4.8. *For K, H, N and G as before and all $x \in A(H)_{(\mathfrak{p})}$, the following identity holds in $A(K)_{(\mathfrak{p})}$:*

$$R^N_K N^N_H(x) = \prod_{n \in K \backslash N / H} N^K_{K \cap {}^{n}H} \cdot c_n \cdot R_{{}^{n^{-1}}K \cap H}(x)$$

where we wrote c_n for the map induced from conjugation by $n \in N$.

Lemma 4.9. *The norms of $A(-)_{(\mathfrak{p})}$ satisfy $\phi^H(N^H_K(a)) = \phi^K(a)$ for all $a \in A(K)_{(\mathfrak{p})}$ and all nested subgroups $K \leq H \leq G$.*

Proof. Let $Q \leq G$ be any subgroup. Under the isomorphism $A(G)_{(\mathfrak{p})} \cong \pi^G_0(\mathbb{S}_{(\mathfrak{p})})$ of Theorem 3.13, the homomorphism of marks $\phi^Q \colon A(G)_{(\mathfrak{p})} \to \mathbb{Z}_{(\mathfrak{p})}$ identifies with the map

$$\pi^G_0(\mathbb{S}_{(\mathfrak{p})}) \to \mathbb{Z}_{(\mathfrak{p})}, \ \left[f \colon \mathbb{S}_{(\mathfrak{p})} \to \mathbb{S}_{(\mathfrak{p})}\right] \mapsto \deg(\phi^Q(f))$$

which sends a class represented by f to the degree of the map $\phi^Q(f)$ induced on geometric fixed points, see [Seg71, p. 60]. Thus, it suffices to prove that the degrees of the two maps of non-equivariant spectra

$$\phi^H\left(\bigwedge_{H/K}\mathrm{Res}_K\mathbb{S}_{(\mathfrak{p})}\to\bigwedge_{H/K}\mathrm{Res}_K\mathbb{S}_{(\mathfrak{p})}\xrightarrow{\cong}\mathrm{Res}_H\mathbb{S}_{(\mathfrak{p})}\right)$$

and $\phi^K(f)$ coincide. This follows immediately from [HHR16, B.209]. □

Corollary 4.10 (Cf. [Oda14], Lemma 2.2)**.** *For $Q, K \le H$, we can compute the marks ϕ^Q of a norm as follows:*

$$\phi^Q N_K^H(x)=\prod_{h\in Q\backslash H/K}\phi^{Q\cap\, {}^hK}(x)$$

Proof. In the following computation, the second equality is the multiplicative double coset formula of Corollary 4.8, and the third uses that ϕ^Q is a ring homomorphism. The fourth equality is an application of Lemma 4.9.

$$\phi^Q N_K^H(x)=\phi^Q R_Q^H N_K^H(x)=\phi^Q\left(\prod_h N^Q_{Q\cap {}^hK}c_h R^K_{h^{-1}Q\cap K}(x)\right)$$

$$=\prod_h\phi^Q N^Q_{Q\cap {}^hK}c_h R^K_{h^{-1}Q\cap K}(x)=\prod_h\phi^{Q\cap {}^hK}c_h R^K_{h^{-1}Q\cap K}(x)=\prod_h\phi^{Q\cap {}^hK}(x)$$

□

Remark 4.11. The analogue of Corollary 4.10 for the complex representation ring $RU(G)$ and the maps which evaluate characters is false. Thus, our results for the splitting of $\mathbb{S}$ do not immediately translate to that of G-equivariant K-theory KU_G, even though the classification of (local) idempotents of $RU(G)$ is very similar to that of $A(G)$.

Lemma 4.12. *For $H \le G$ and $g \in G$, the following holds:*

a) $O^{\mathfrak{p}}(H)\subseteq O^{\mathfrak{p}}(G)$
b) $O^{\mathfrak{p}}({}^gH)={}^g(O^{\mathfrak{p}}(H))$

The author learned the proof of part a) from Joshua Hunt.

Proof. Since $O^{\mathfrak{p}}(G)$ is normal in G, we know that $H\cap O^{\mathfrak{p}}(G)$ is normal in H. Now the group $H/(H\cap O^{\mathfrak{p}}(G))\cong(H\cdot O^{\mathfrak{p}}(G))/O^{\mathfrak{p}}(G)\le G/O^{\mathfrak{p}}(G)$ is isomorphic to a subgroup of a solvable $\mathfrak{p}$-group, hence is a solvable $\mathfrak{p}$-group itself. By minimality, $O^{\mathfrak{p}}(H)\le$

$H \cap O^{\mathfrak{p}}(G) \leq O^{\mathfrak{p}}(G)$, which proves a).
The assertion b) follows from the fact that conjugation by g induces a bijection between the subgroup lattices of H and ${}^{g}H$ which preserves normality. □

We will now use these technical results to translate the division relation stated in Theorem 2.33 into the group-theoretic condition ($\diamond$) given in the following result. Finally, we explain how to prove Theorem 4.1 from it.

Theorem 4.13. *In the situation of Theorem 4.1, the norm N_K^H descends to a well-defined map of multiplicative monoids*

$$\tilde{N}_K^H \colon A(K)_{(\mathfrak{p})}[e^{-1}] \to A(H)_{(\mathfrak{p})}[e^{-1}]$$

if and only if the following holds:

($\diamond$) *For all subgroups $Q \leq H$ such that $O^{\mathfrak{p}}(Q)$ is conjugate in G to L, the group L is subconjugate in G to $Q \cap K$.*

Proof. Fix an H-set H/K. We know from Theorem 2.33 that the norm N_K^H descends to a well-defined $\tilde{N}_K^H$ if and only if a certain division relation holds, which is stated below as (I). We now show that the following statements are all equivalent.

(I) The element $N_K^H R_K^G(e_L)$ divides $R_H^G(e_L)$ in $A(H)_{(\mathfrak{p})}$.
(IIa) For all $Q \leq H$, the element $\phi^Q(N_K^H R_K^G(e_L))$ divides $\phi^Q(R_H^G(e_L))$ in $\mathbb{Z}_{(\mathfrak{p})}$.
(IIb) For all $Q \leq H$ such that $\phi^Q(R_H^G(e_L)) = 1$, we have $\phi^Q(N_K^H R_K^G(e_L)) = 1$.
(IIc) For all $Q \leq H$ such that $O^{\mathfrak{p}}(Q) \sim_G L$, we have $O^{\mathfrak{p}}(Q \cap {}^{h}K) \sim_G L$ for all $h \in H$.
(IId) For all $Q \leq H$ such that $O^{\mathfrak{p}}(Q) \sim_G L$, we have $L \leq_G O^{\mathfrak{p}}(Q \cap {}^{h}K)$ for all $h \in H$.
(IIe) For all $Q \leq H$ such that $O^{\mathfrak{p}}(Q) \sim_G L$, we have $L \leq_G Q \cap {}^{h}K$ for all $h \in H$.
($\diamond$) For all $Q \leq H$ such that $O^{\mathfrak{p}}(Q) \sim_G L$, we have $L \leq_G Q \cap K$.

(I) $\Leftrightarrow$ (IIa): This equivalence holds since the homomorphism of marks

$$\phi = \prod_{(Q) \leq H} \phi^Q \colon A(H)_{(\mathfrak{p})} \to \prod_{(Q) \leq H} \mathbb{Z}_{(\mathfrak{p})}$$

is an injective ring homomorphism.
(IIa) $\Leftrightarrow$ (IIb): Each of the two elements in the division relation has to be either 0 or 1 because these are the only idempotents in the ring $Z_{(\mathfrak{p})}$.
(IIb) $\Leftrightarrow$ (IIc): We can translate the two equations in (IIb) into group-theoretic statements, using the description of the marks of a primitive idempotent given in Theorem 3.4, and applying the formula for the marks of a norm from Corollary 4.8. The left-hand equation holds if and only if $O^{\mathfrak{p}}(Q)$ is G-conjugate to L, whereas the right-hand

equation holds if and only if $O^{\mathfrak{p}}(Q \cap {}^{h}K)$ is G-conjugate to L for all $h \in H$.
(IIc) $\Leftrightarrow$ (IId): For $Q \leq H$ such that $O^{\mathfrak{p}}(Q) = {}^{g^{-1}}L$ for some $g \in G$, Lemma 4.12 implies

$$O^{\mathfrak{p}}(Q \cap {}^{h}K) \leq O^{\mathfrak{p}}(Q) \cap O^{\mathfrak{p}}({}^{h}K) \sim_G L \cap {}^{g}(O^{\mathfrak{p}}({}^{h}K)) \leq L,$$

hence $O^{\mathfrak{p}}(Q \cap {}^{h}K)$ is always G-conjugate to a subgroup of L.
(IId) $\Leftrightarrow$ (IIe): This follows easily from Lemma 4.12.
(IIe) $\Leftrightarrow$ ($\diamond$): The implication (IIe) $\Rightarrow$ ($\diamond$) is obvious. For the converse, fix $h \in H$ and note that the group $Q' := {}^{h^{-1}}Q$ satisfies the hypothesis of ($\diamond$) because the groups $O^{\mathfrak{p}}({}^{h^{-1}}Q), O^{\mathfrak{p}}(Q)$ and L are all conjugate in G. We thus obtain

$$L \leq_G Q' \cap K = {}^{h^{-1}}Q \cap K \sim_G {}^{h}({}^{h^{-1}}Q \cap K) = Q \cap {}^{h}K.$$

To complete the proof of the theorem, we observe that if a well-defined norm map $\tilde{N}_K^H$ exists, it is automatically unital and multiplicative, as follows from Formula 2.34. □

Proof of Theorem 4.1. It suffices to show that the condition ($\diamond$) of Theorem 4.13 is equivalent to the conditions given in Theorem 4.1. Three cases arise for $K \leq H$:

i) Neither K and H are super-conjugate in G to L: In this case there is no group Q satisfying the hypothesis of ($\diamond$) in Theorem 4.13, hence ($\diamond$) is trivially satisfied and the norm $\tilde{N}_K^H$ exists.
ii) If only H is super-conjugate in G to L but K is not, then there is no norm map $\tilde{N}_K^H$. Choose $Q := L$, then the condition ($\diamond$) is not satisfied, as L is contained in H but not in K.
iii) Consider the case where both K and H are super-conjugate to L. Clearly, the condition ($\diamond$) implies the condition ($\star$) from Theorem 4.1, and we have to prove the converse. To that end, observe that when $Q \leq Q' \leq H$ such that Q satisfies ($\diamond$), then so does Q'. Thus, we can reduce the collection of groups Q for which ($\diamond$) has be to tested: ($\diamond$) holds for all groups in the poset $\{Q \leq H \mid O^{\mathfrak{p}}(Q) \sim_G L\}$ if and only if it holds for the minimal elements. But these are exactly the conjugate copies L' of L which are contained in H. Such a group L' is contained in K if and only if L is subconjugate to $L' \cap K$. □

4.3. **The incomplete Tambara functor structure.** It still remains to see how the collection of norm maps described by Theorem 4.1 fits into the framework of [BH16]. First of all, we describe the norm maps in $A(-)_{(\mathfrak{p})}[e_L^{-1}]$ arising from arbitrary maps of G-sets. This is the special case $\underline{R} = A(-)_{(\mathfrak{p})}$, $x = e_L$ of the following result:

Proposition 4.14. *Let $x \in \underline{R}(G)$ be an idempotent. Let f be an arbitrary map of finite G-sets. Choose orbit decompositions of X and Y such that f is the sum of canonical surjections*

$$f\colon X = \coprod_{i,j} G/K_{ij} \to Y = \coprod_i G/H_i$$

induced by subgroup inclusions $K_{ij} \le H_i$. Then the levelwise localization $\underline{R}[x^{-1}]$ inherits a norm map $\tilde{N}_f$ from $\underline{R}$ if and only if each restriction to orbits $f_{ij}\colon G/K_{ij} \to G/H_i$ does.

Proof. The proof proceeds in two steps.
Step 1: By the universal property of the product (of underlying multiplicative monoids), a potential norm map defined by f is given componentwise by the potential norms induced by the restricted maps $f_i\colon \coprod_j G/K_{ij} \to G/H_i$. Consequently, $\tilde{N}_f$ exists if and only if $\tilde{N}_{f_i}$ exists for all i.
Step 2: We are left to show that a map $f_i\colon \coprod_j G/K_{ij} \to G/H_i$ gives rise to a norm map if and only if all of the maps $f_{ij}\colon G/K_{ij} \to G/H_i$ do. But under the identification $\underline{R}(\coprod_j G/K_{ij}) \cong \prod_j \underline{R}(K_{ij})$, the norm N_f is of the form

$$\prod_j \underline{R}(K_{ij}) \to \underline{R}(H_i), \quad (a_j)_j \mapsto \prod_j N_{f_{ij}}(a_j).$$

The analogous statement holds for the norms of $\underline{R}[x^{-1}]$, provided they exist. Thus, $\tilde{N}_{f_i}$ exists if and only if $\tilde{N}_{f_{ij}}$ exists for all j. $\square$

We would like to use Theorem 2.33 in order to show that $A(-)_{(\mathfrak{p})}[e_L^{-1}]$ is an incomplete Tambara functor with norms as described in Theorem 4.1. However, Theorem 2.33 is a statement about Tambara functors structured by indexing systems, or equivalently (see Theorem 2.15), structured by wide, pullback-stable, finite coproduct-complete subcategories $D \subseteq \mathcal{S}et^G$. Thus, we first need to see that the maps f which give rise to norm maps form such a category D.

Definition 4.15. Let $D_L \subseteq \mathcal{S}et^G$ be the wide subgraph consisting of all the maps of finite G-sets $f\colon X \to Y$ such that the orbit $G_{f(x)}/G_x$ obtained from stabilizer subgroups satisfies the conditions of Theorem 4.1 for all $x \in X$.

Proposition 4.16. *The subgraph D_L is a wide, pullback-stable, finite coproduct-complete subcategory of $\mathcal{S}et^G$, hence corresponds to an indexing system $\mathcal{I}_L$ under the equivalence of posets of Theorem 2.15.*

Explicitly, the admissible H-sets in $\mathcal{I}_L$ are the objects over G/H in D_L, see [BH16, Lemma 3.18]. The three lemmas below constitute the proof.

Lemma 4.17. *The graph D_L is a wide subcategory of $\mathcal{S}et^G$.*

Proof. It is wide by definition and clearly contains all identities. Once we have shown that it is closed under composition, associativity follows from associativity in $\mathcal{S}et^G$. Let $f\colon S \to T$ and $g\colon T \to U$ be admissible maps of G-sets. By Proposition 4.14, we may assume that $S = G/A, T = G/B$ and $U = G/C$ are transitive G-sets for nested subgroups $A \leq B \leq C \leq G$, and f, g are the canonical surjections. Thus, it suffices to show that if C/B and B/A are admissible, so is C/A. This is immediate in either of the two cases of Theorem 4.1. □

Lemma 4.18. *The subcategory D_L is finite coproduct-complete.*

Proof. This follows directly from Proposition 4.14. □

Lemma 4.19. *The subcategory D_L is pullback-stable.*

Proof. The problem reduces to canonical surjections between orbits by Proposition 4.14. We have to show that if the canonical surjection $G/K \to G/H$ in the following pullback diagram is admissible, then so is its pullback along the canonical map $G/A \to G/H$, where $A, K \leq H$ are subgroups.

$$\begin{array}{ccc} P & \longrightarrow & G/K \\ \downarrow & & \downarrow \\ G/A & \longrightarrow & G/H \end{array}$$

This in turn amounts to verifying the conditions of Theorem 4.1 for all summands of

$$R^H_A(H/K) \cong \coprod_{[h]\in A\backslash H/K} A/(A \cap {}^hK).$$

If L is not subconjugate in G to A, then each $A/(A \cap {}^hK)$ is admissible by case i) of Theorem 4.1. So assume that L is subconjugate in G to A. In particular, L is subconjugate to H. Since H/K is admissible, Theorem 4.1 implies that L must be subconjugate to K. Thus, A is subconjugate to $A \cap {}^hK$ for any $h \in H$ and it remains to check condition ($\star$) for $A \cap {}^hK \leq A$. Let L' be a conjugate of L such that $L' \leq A \leq H$. Since H/K is admissible, so is $H/{}^hK$, and Theorem 4.1 implies that $L' \leq {}^hK$. Hence, $L' \leq A \cap {}^hK$ which shows that ($\star$) holds for $A/A \cap {}^hK$ and any $h \in H$. □

We obtain (the locally enhanced) Theorem D:

Theorem 4.20. *Let $\mathfrak{p}$ be a collection of primes. Let $L \leq G$ be a $\mathfrak{p}$-perfect subgroup and let $e_L \in A(G)_{(\mathfrak{p})}$ be the corresponding primitive idempotent. Then the following hold:*

i) The admissible sets for e_L assemble into an indexing system $\mathcal{I}_L$ such that $A(-)_{(\mathfrak{p})}[e_L^{-1}]$ is an $\mathcal{I}_L$-Tambara functor under $A(-)_{(\mathfrak{p})}$.

ii) In the poset of indexing systems, $\mathcal{I}_L$ is maximal among the elements that satisfy i).

iii) The map $A(-)_{(\mathfrak{p})} \to A(-)_{(\mathfrak{p})}[e_L^{-1}]$ is the localization of $A(-)_{(\mathfrak{p})}$ at e_L in the category of $\mathcal{I}_L$-Tambara functors.

Proof. Proposition 4.16 shows that $\mathcal{I}_L$ is an indexing system. Then $A(-)_{(\mathfrak{p})}[e_L^{-1}]$ is an $\mathcal{I}_L$-Tambara functor by [BH16, Thm. 4.13], see the proof of Theorem 2.33 for details. Theorem 2.33 also implies part iii). Part ii) follows from Theorem 4.1 together with Proposition 4.14. □

Finally, we describe the maximal incomplete Tambara functor structure which is preserved by the idempotent splitting of the Green ring $A(-)_{(\mathfrak{p})}$ stated in Proposition 3.9.

Lemma 4.21. *The (levelwise) intersection of a finite number of indexing systems is an indexing system.*

Proof. This is clear from the definition, see Definition 2.5. □

Notation 4.22. Write $\mathcal{I}$ for the indexing system

$$\mathcal{I} := \bigcap_{(L)\leq G} \mathcal{I}_L$$

where the intersection is over all conjugacy classes of $\mathfrak{p}$-perfect subgroups of G, and the indexing systems $\mathcal{I}_L$ are the ones given by Theorem 4.20.

For each $\mathfrak{p}$-perfect $L \leq G$, the $\mathcal{I}_L$-Tambara functor $A(-)_{(\mathfrak{p})}[e_L^{-1}]$ is an $\mathcal{I}$-Tambara functor by forgetting structure. Theorem 4.1 provides a very explicit description of the admissible sets of $\mathcal{I}$.

Lemma 4.23. *Let $K \leq H \leq G$, then H/K is an admissible set for $\mathcal{I}$ if and only if for all perfect $L \leq H$, L is contained in K.*

We can now restate Corollary E.

Corollary 4.24. *The localization maps $A(-)_{(\mathfrak{p})} \to A(-)_{(\mathfrak{p})}[e_L^{-1}]$ assemble into an isomorphism of $\mathcal{I}$-Tambara functors*

$$A(-)_{(\mathfrak{p})} \to \prod_{(L)\leq G\ \mathfrak{p}-\text{perfect}} A(-)_{(\mathfrak{p})}[e_L^{-1}].$$

Proof. It is an isomorphism of Green rings by Proposition 3.9. Moreover, each of the localization maps $A(-)_{(\mathfrak{p})} \to A(-)_{(\mathfrak{p})}[e_L^{-1}]$ is a map of $\mathcal{I}$-Tambara functors, and the product in the category of $\mathcal{I}$-Tambara functors is computed levelwise, see [Str12, Prop. 10.1]. □

Remark 4.25. We point out a possible alternative to our proof of Corollary 4.24. Blumberg and Hill generalized parts of Nakaoka's theory of ideals of Tambara functors [Nak12] to the setting of incomplete Tambara functors, see [BH16, Section 5.2]. The author is confident that one could similarly generalize Nakaoka's splitting result [Nak12, Prop. 4.15] to the incomplete setting. It would state that an $\mathcal{I}$-Tambara functor $\underline{R}$ splits non-trivially as a product of $\mathcal{I}$-Tambara functors if and only if for each admissible set X of $\mathcal{I}$, there are non-zero elements $a, b \in \underline{R}(X)$ such that $a + b = 1$ and $\langle a \rangle \cdot \langle b \rangle = 0$. Such a result would reprove our Corollary 4.24, using that the restrictions of the primitive idempotents e_L along admissible maps never become zero. We leave the details to the interested reader.

4.4. **The N_∞ ring structure.** We return to the situation of Question 1.2, lift our algebraic results to the category of G-spectra and prove the locally enhanced versions of Corollary F, Corollary G and Corollary H.

Corollary 4.26. *Let $L \leq G$ be a $\mathfrak{p}$-perfect subgroup and let $e_L \in \underline{\pi}_0^G(\mathbb{S})$ be the associated idempotent. For any Σ-cofibrant N_∞ operad $\mathcal{O}_L$ whose associated indexing system is $\mathcal{I}_L$, the following hold:*

i) The G-spectrum $\mathbb{S}_{(\mathfrak{p})}[e_L^{-1}]$ is (canonically equivalent to) an $\mathcal{O}_L$-algebra under $\mathbb{S}_{(\mathfrak{p})}$.
ii) In the poset of homotopy types of N_∞ operads, $\mathcal{O}_L$ is maximal among the elements that satisfy i).
iii) The map $\mathbb{S}_{(\mathfrak{p})} \to \mathbb{S}_{(\mathfrak{p})}[e_L^{-1}]$ is a localization at e_L in the category of $\mathcal{O}_L$-algebras.

The cofibrancy assumption does not impose an obstruction to the existence of $\mathcal{O}_L$, see Remark 2.11.

Proof. For any such operad $\mathcal{O}_L$, the G-homotopy equivalence $\mathcal{O}_L(0) \to *$ induces a canonical homotopy equivalence of G-spectra $\Sigma^\infty_+ \mathcal{O}_L(0) \simeq \mathbb{S}$, thus $\mathbb{S}_{(\mathfrak{p})}$ identifies with the $\mathcal{O}_L$-algebra $(\Sigma^\infty_+ \mathcal{O}_L(0))_{(\mathfrak{p})}$. It is clear from Theorem 2.33 that $\mathcal{O}_L$ satisfies the hypothesis of Proposition 2.26, which proves part i) and iii). For part ii), assume that there is an N_∞ operad $\mathcal{O}'$ whose homotopy type is strictly greater than that of $\mathcal{O}_L$ such that $\mathbb{S}_{(\mathfrak{p})}[e_L^{-1}]$ is an $\mathcal{O}'$-algebra (up to equivalence). Then, by Theorem 2.20, its zeroth equivariant homotopy forms a $\mathcal{I}'$-Tambara functor for the indexing system $\mathcal{I}'$ corresponding to $\mathcal{O}'$. But this contradicts the maximality proved in Corollary 4.20. □

The following local enhancement of Corollary G is a homotopical reformulation of Corollary 4.3 (Corollary C).

Corollary 4.27. *The G-spectrum $\mathbb{S}_{(\mathfrak{p})}[e_L^{-1}]$ is a G-E_∞ ring spectrum if and only if $L = 1$ is the trivial group.*

In particular, we see that the idempotent splitting of $\mathbb{S}$ is far from being a splitting of G-E_∞ ring spectra. Locally at the prime p, Corollary 4.27 recovers a (yet unpublished) result of Grodal.

Theorem 4.28 ([Gro], Cor. 5.5)**.** *The G-spectrum $\mathbb{S}_{(p)}[e_1^{-1}]$ is a G-E_∞ ring spectrum.*

Finally, we state the homotopy-theoretic analogue of Corollary 4.24 in order to describe the maximal N_∞-ring structure preserved by the $\mathfrak{p}$-local idempotent splitting of the sphere. It is the local reformulation of Corollary H.

Corollary 4.29. *Let $\mathcal{O}$ be a Σ-cofibrant N_∞ operad realizing the indexing system $\mathcal{I} = \cap_{(L)}\mathcal{I}_L$. Under the identification $(\Sigma^\infty_+\mathcal{O}(0))_{(\mathfrak{p})} \simeq \mathbb{S}_{(\mathfrak{p})}$, the idempotent splitting*

$$\mathbb{S}_{(\mathfrak{p})} \simeq \prod_{(L)\leq G} \mathbb{S}_{(\mathfrak{p})}[e_L^{-1}]$$

is an equivalence of $\mathcal{O}$-algebras, where the product is taken over conjugacy classes of $\mathfrak{p}$-perfect subgroups.

Proof. The splitting is an equivalence of G-spectra by Proposition 3.15. Moreover, under the above identifications, all of the maps to the localizations are maps of $\mathcal{O}$-algebras, as can be seen from 4.26. □

Together, Corollary 4.26 and Corollary 4.29 answer Question 1.2 completely, for any family of primes inverted.

5. Examples

We illustrate our results in the rational case, in the case of the alternating group A_5, working integrally, and that of the symmetric group Σ_3, working 3-locally.

5.1. **The rational case.** In the case when $\mathfrak{p} = \emptyset$ and hence $\mathbb{Z}_{(\mathfrak{p})} = \mathbb{Q}$, the rational Burnside ring $A(G)_{\mathbb{Q}}$ has exactly one primitive idempotent e_L for each conjugacy class of subgroups $L \leq G$. The incomplete Tambara functor structures of the idempotent summands $A(-)_{\mathbb{Q}}[e_L^{-1}]$ depend on the subgroup structure of G as described by Theorem 4.1. However, it is immediately clear from Lemma 4.23 that the idempotent splitting is only a splitting of Green rings, but not a splitting of $\mathcal{I}'$-Tambara functors for any indexing system $\mathcal{I}'$ greater than the minimal one. This phenomenon is also discussed in [BGK17, Section 7], and it is precisely the reason why their approach only provides an algebraic model for the rational homotopy theory of naive N_∞ ring spectra, but cannot possibly account for any non-trivial Hill-Hopkins-Ravenel norms.

5.2. **The alternating group A_5.** It is well-known that A_5 is the smallest non-trivial perfect group. Thus, it is the smallest example of a group whose Burnside ring admits a non-trivial idempotent splitting when working integrally. Indeed, the only perfect subgroups are 1 and A_5, and these give rise to idempotent elements $e_1, e_{A_5} \in A(A_5)$. Theorem 3.4 implies that their marks are given by

$$\phi^H(e_{A_5}) = \begin{cases} 1, & H = A_5 \\ 0, & H \neq A_5 \end{cases}$$

and vice versa for e_1. We know from Corollary 4.3 that $A(A_5)[e_1^{-1}]$ is a complete Tambara functor. On the other hand, $A(H)[e_{A_5}^{-1}]$ is trivial unless $H = A_5$, hence there cannot be any norm maps $N_H^{A_5}$ for proper subgroups $H \leq A_5$. Moreover, by Corollary 4.24, the idempotent splitting of $A(-)$ is a splitting of $\mathcal{I}_{A_5}$-Tambara functors, i.e., it only preserves norms between proper subgroups.
By Corollary 4.26 and Corollary 4.29, the analogous statements hold for the N_∞ ring structures on $\mathbb{S}[e_1^{-1}]$ and $\mathbb{S}[e_{A_5}^{-1}]$. Just like Example 2.23, this provides another instance of the phenomenon that inverting a single homotopy element does not preserve any of the Hill-Hopkins-Ravenel norm maps from proper subgroups to the ambient group.
Of course, all of this holds for any perfect group G whose only perfect subgroup is the trivial group.

5.3. **The symmetric group Σ_3 at the prime 3.** Since Σ_3 is solvable, its Burnside ring $A(\Sigma_3)$ does not have any idempotents other than zero and one. We can obtain interesting idempotent splittings by working locally at primes p dividing the group order. All 2-perfect subgroups of Σ_3 are normal, hence the case $p = 2$ is completely covered by Corollary 4.2 and we only discuss the more interesting case $p = 3$ in detail.

Any map in the orbit category can be factored as an isomorphism followed by a canonical surjection, hence the admissibilty of Σ_3/H just depends on the conjugacy class of

H and we can just write C_2 for any of the three conjugate subgroups of order two. Note that the 3-residual subgroups $O^3(H)$ for $H \leq \Sigma_3$ are given as follows:

$$\mathrm{O}^3(H) = \begin{cases} \Sigma_3, & H = \Sigma_3 \\ 1, & H = A_3 \\ C_2, & H = C_2 \\ 1, & H = 1 \end{cases}$$

Thus, all subgroups of Σ_3 except for A_3 are 3-perfect. All subgroups of order two are conjugate in Σ_3, so there are three idempotent elements in $A(\Sigma_3)_{(3)}$, corresponding to the conjugacy classes of the 3-perfect subgroups $1, C_2$ and Σ_3. In terms of marks, they are given as

Subgroup $H \leq \Sigma_3$	$\phi^H(e_1)$	$\phi^H(e_{C_2})$	$\phi^H(e_{\Sigma_3})$
1	1	0	0
C_2	0	1	0
A_3	1	0	0
Σ_3	0	0	1

The localization $A(-)_{(3)}[e_1^{-1}]$ admits all norms by Corollary 4.3. The norm maps of $A(-)_{(3)}[e_{\Sigma_3}^{-1}]$ are described by Corollary 4.2. In detail, this Mackey functor is zero at all proper subgroups, but non-trivial at Σ_3. Consequently, there are norm maps $\tilde{N}_K^H$ if and only if H and hence K is a proper subgroup of Σ_3, but all of these norms are maps between trivial rings.

It remains to describe the idempotent localization $A(-)_{(3)}[e_{C_2}^{-1}]$. The left of the following two diagrams depicts the subgroups $H \leq \Sigma_3$ (up to conjugacy) and their inclusions. The right hand diagram displays the ranks (as free $\mathbb{Z}_{(3)}$-modules) of the corresponding values of the incomplete Tambara functor $A(-)_{(3)}[e_{C_2}^{-1}]$ at the subgroup H.

$$\begin{tikzcd} & \Sigma_3 & \\ & & A_3 \arrow[ul] \\ C_2 \arrow[uur] & & \\ & 1 \arrow[uuu] \arrow[uur] \arrow[ul] & \end{tikzcd} \qquad\qquad \begin{tikzcd} & 1 & \\ & & 0 \\ 1 \arrow[uur, dashed] & & \\ & 0 \arrow[uur] & \end{tikzcd}$$

There is a norm map from 1 to A_3 for trivial reasons (indicated by the solid arrow) and the only other norm maps which could potentially exist would be the maps $\tilde{N}_{C_2}^{\Sigma_3}$ where C_2 is any subgroup of order two (indicated by the dashed arrow). However, if we choose $K = L = (12)$ and let $L' = (13)$, then condition $(\star)$ in Theorem 4.1 is not satisfied. Indeed, L' is conjugate to L, but not contained in K. Consequently, there is no norm map $\tilde{N}_{C_2}^{\Sigma_3}$.
We see from Lemma 4.23 that $\mathcal{I} = \mathcal{I}_{C_2}$, so in this case the idempotent splittings of $A(-)_{(3)}$ and hence $\mathbb{S}_{(3)}$ only preserve the norm map $N_1^{A_3}$.

6. Applications and future directions

6.1. **Norm functors in the idempotent splitting of** Sp^G**.** Any G-spectrum X is a module over the sphere spectrum, hence admits an idempotent splitting

$$X \simeq \prod_{(L) \leq G} X[e_L^{-1}]$$

where $X[e_L^{-1}]$ is the sequential homotopy colimit along countably many copies of the map $X \cong X \wedge \mathbb{S} \xrightarrow{\mathrm{id} \wedge e_L} X \wedge \mathbb{S} \cong X$. Thus, the idempotent elements of $A(G)$ induce a splitting of the category of G-spectra Sp^G, and similar statements hold in the local cases, cf. e.g. [Bar09, Thm. 4.4, Section 6]. While this only depends on the additive splitting of Proposition 3.15, some additional multiplicative structure is present.

It is useful to consider not just the category of ($\mathfrak{p}$-local) G-spectra, but rather the symmetric monoidal categories of ($\mathfrak{p}$-local) H-spectra for all subgroups $H \leq G$ together with their restriction and norm functors. This kind of structure has been studied in [HH16, BH15a] under the name of *G-symmetric monoidal categories*. From this perspective, Theorem A measures the failure of the idempotent splitting of Sp^G to give rise to a splitting of G-symmetric monoidal categories. Indeed, the factors only admit some of the Hill-Hopkins-Ravenel norm functors and hence form "incomplete G-symmetric monoidal categories":

Corollary 6.1. *Let $L \leq G$ be $\mathfrak{p}$-perfect and let $\mathcal{O}_L$ as in Corollary 4.26. Assume furthermore that $\mathcal{O}_L$ has the homotopy type of the linear isometries operad on a (possibly incomplete) universe U. For all admissible sets H/K of $\mathcal{I}_L$, there are norm functors relative to $\mathbb{S}_{(\mathfrak{p})}[e_L^{-1}]$*

$${}_{\mathrm{Res}_H(\mathbb{S}_{(\mathfrak{p})}[e_L^{-1}])}N_{K,\mathrm{Res}_K(U)}^{H,\mathrm{Res}_H(U)} : Mod(\mathrm{Res}_K^G(\mathbb{S}_{(\mathfrak{p})}[e_L^{-1}])) \to Mod(\mathrm{Res}_H^G(\mathbb{S}_{(\mathfrak{p})}[e_L^{-1}]))$$

which on the level of homotopy categories form the structure of an "incomplete Mackey functor in symmetric monoidal categories".

This is an immediate application of [BH15a, Thm. 1.1, Thm. 1.3] to Corollary 4.26. We refer to [BH15a] for a detailed discussion of modules over N_∞ ring spectra.

The reason for the "linear isometries" hypothesis is explained in the introduction to [BH15a]. It is expected that it is not necessary, and that the ∞-categorical tools developed in [BDG+17] and its sequels will remove this technical assumption.

6.2. **Idempotent splittings of equivariant K-theory and other G-E_∞-rings.** Our main question, Question 1.2, can be asked for any G-E_∞ ring spectrum and its idempotent splitting. The most accessible such object seems to be the G-equivariant complex K-theory spectrum KU_G, for reasons which we explain now.

First observe that $\underline{\pi}_0(KU_G)$ is the complex representation ring Tambara functor $RU(-)$. For any finite group H, $RU(H)$ is connected, i.e., does not admit any idempotents other than zero and one. However, interesting splittings exist locally, and the classification of idempotent elements in the p-local representation ring $RU(G)_{(p)}$ is quite similar to that of $A(G)_{(p)}$: they correspond to conjugacy classes of group elements whose order is prime to p. Thus, it is natural to ask whether the program of the present paper can be carried out with $\mathbb{S}$ replaced by the equivariant ring spectrum KU_G. However, the proof of Theorem A does not immediately carry over, see Remark 4.11. We intend to return to this question in future work.

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