ON THE HOMOTOPY THEORY OF K-LOCAL SPECTRA AT AN ODD PRIME

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Introduction. In this paper we investigate the algebraic structure of the stable homotopy category localized with respect to K-homology theory at an odd prime, and we give a purely algebraic classification of all homotopy types in that localized category.

Before describing our results more fully, we must recall the theory of homological localizations of spectra (see [5], [7], [11]). Let $\underline{Ho}^s$ denote the homotopy category of CW-spectra (see [5]) or any of the other equivalent versions of Boardman's stable homotopy cateogry. A spectrum $E \in \underline{Ho}^s$ determines a homology theory E_* with $E_*X = \pi_*E \wedge X$ for each $X \in \underline{Ho}^s$, and a spectrum $Y \in \underline{Ho}^s$ is called *E_*-local* if each E_*-equivalence $A \to B$ in $\underline{Ho}^s$ induces an isomorphism $[B, Y]_* \approx [A, Y]_*$. For each $E \in \underline{Ho}^s$ there is a natural E_*-localization which assigns to each spectrum $X \in \underline{Ho}^s$ an E_*-equivalence $X \to X_E$ in $\underline{Ho}^s$ such that X_E is E_*-local. It follows that the full subcategory of E_*-local spectra in $\underline{Ho}^s$ is equivalent to the category of fractions obtained from $\underline{Ho}^s$ by giving formal inverses to the E_*-equivalences. In this paper, we are interested in the case $E = K_{(p)}$ for an odd prime p where $K_{(p)}$ is the spectrum of nonconnective complex K-theory localized at p. However, we often find it convenient to replace $K_{(p)}$ by its summand $E(1)$ from the well-known splitting $K_{(p)} \simeq \vee_{i=0}^{p-2} \Sigma^{2i}E(1)$ (see Section 4). Since the $K_{(p)*}$-equivalences in $\underline{Ho}^s$ are the same as the $E(1)_*$-equivalences, it follows that the $K_{(p)*}$-localization in $\underline{Ho}^s$ is the same as the $E(1)_*$-localization. This localization has previously been studied by Adams-Baird (unpublished), Ravenel [11], and the author [7]. We remark that it is the next after the rational localization in Ravenel's hierarchy of localizations which take account of progressively higher sorts of periodicity phenomena in p-local stable homotopy theory.

In order to approach the homotopy theory of a $K_{(p)*}$-local ($= E(1)_*$-local) spectrum X, we first consider the homology groups $K_{(p)*}X$ or

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$E(1)_*X$ together with appropriate primary operations. The required information can be captured by treating $K_{(p)*}X$ or $E(1)_*X$ as a comodule over the coalgebra $K_{(p)*}K_{(p)}$ or $E(1)_*E(1)$. However, this comodule structure is somewhat awkward to work with, and we find that exactly the same information is more conveniently captured by the stable Adams operations ψ^k with k prime to p. To formalize this, we construct graded abelian categories $\mathcal{A}(p)_*$ and $\mathcal{B}(p)_*$ whose objects have the algebraic properties of the homology groups $K_{(p)*}X$ and $E(1)_*X$ respectively with their stable Adams operations. We show that $\mathcal{A}(p)_*$ and $\mathcal{B}(p)_*$ are canonically equivalent to each other and we use $\mathcal{B}(p)_*$ in most of our subsequent work. In the last section (Section 10) we demonstrate that $\mathcal{A}(p)_*$ and $\mathcal{B}(p)_*$ are canonically equivalent to the categories of $K_{(p)*}K_{(p)}$-comodules and $E(1)_*E(1)$-comodules respectively. This is of independent interest and generalizes a somewhat similar equivalence obtained by Ravenel [11] for torsion $E(1)_*E(1)$-comodules using a very different approach. In Sections 1, 3 we construct $\mathcal{A}(p)_*$ and $\mathcal{B}(p)_*$ as graded periodic versions of ungraded abelian categories $\mathcal{A}(p)$ and $\mathcal{B}(p)$ which resemble the abelian category of Adams [1] but differ because we impose rational diagonalizability conditions on our operations and work in a p-local nonfinitely generated context. In Section 5 we give a very simple alternative construction of $\mathcal{B}(p)$ involving a single operation ψ^r, and we see that a torsion object of $\mathcal{B}(p)$ is merely a p-torsion abelian group with a locally nilpotent operator. Interestingly enough, as noted by Ravenel [11], these same torsion objects have been studied by Iwasawa in connection with cyclotomic fields (see [8], [9], [12]). The structure of $\mathcal{B}(p)_*$ is likewise very accesible since that category is equivalent to the product of $2p - 2$ copies of $\mathcal{B}(p)$. In Section 7 we obtain detailed results on the groups $\mathrm{Ext}^{s,t}_{\mathcal{B}(p)_*}(M, N)$ for $s = 0, 1, 2$ and show that they all vanish for $s > 2$ so that the objects of $\mathcal{B}(p)_*$ have injective dimension ≤ 2. A slightly weaker version of this vanishing result was obtained by Adams-Baird in the equivalent category of $K_{(p)*}K_{(p)}$-comodules (see [4]).

In order to investigate the homotopy theory of $E(1)_*$-local ($= K_{(p)*}$-local) spectra, we need the $E(1)_*$-Adams spectral sequence. In Section 8 we construct a version which has

$$E_2^{s,t}(X, Y) \approx \mathrm{Ext}^{s,t}_{\mathcal{B}(p)_*}(E(1)_*X, E(1)_*Y)$$

for arbitrary $X, Y \in \underline{Ho}^s$ and converges strongly to $[X_{E(1)}, Y_{E(1)}]_*$. We next note that $E_2^{s,t}(X, Y) = 0$ for $s > 2$, so that the only possible nontrivial differential is

$$d_2 : \mathrm{Hom}_{\mathcal{B}(p)_*}(E(1)_*X, E(1)_*Y)_t \to \mathrm{Ext}^{2,t+1}_{\mathcal{B}(p)_*}(E(1)_*X, E(1)_*Y).$$

To determine this d_2 algebraically, we introduce the canonical $E(1)_*$-k-*invariant*

$$k_W \in \mathrm{Ext}^{2,1}_{\mathcal{B}(p)_*}(E(1)_*W, E(1)_*W)$$

for each spectrum $W \in \underline{Ho}^s$ and prove the general formula $d_2 f = k_Y \circ f + (-1)^{t+1} f \circ k_X$. Our definition of k_W involves the theory of $E(1)_*$-Moore spectra. For each object $G \in \mathcal{B}(p)$ and each $n \in Z$, we build an $E(1)_*$-local spectrum $\mathfrak{M}(G, n)$ with an isomorphism $E(1)_n \mathfrak{M}(G, n) \approx G$ in $\mathcal{B}(p)$ and with $E(1)_i \mathfrak{M}(G, n) = 0$ for $i \not\equiv n \bmod 2p - 2$. We call $\mathfrak{M}(G, n)$ an $E(1)_*$-*Moore spectrum* and show that it is unique up to a canonical equivalence and depends functorially on $G \in \mathcal{B}(p)$. Also for each object $H \in \mathcal{B}(p)_*$ we build a natural $E(1)_*$-local spectrum $\mathfrak{M}(H) = \vee_{n=0}^{2p-3} \mathfrak{M}(H_n, n)$ with $E(1)_* \mathfrak{M}(H) \approx H$ in $\mathcal{B}(p)_*$, and we call $\mathfrak{M}(H)$ a *generalized* $E(1)_*$-*Moore spectrum*. For $W \in \underline{Ho}^s$, our $E(1)_*$-k-invariant k_W measures the obstruction to finding a map $W \to \mathfrak{M}(E(1)_* W)$ inducing the identity on $E(1)_* W$.

In Section 9 we arrive at our main results on the algebraic classification of the $E(1)_*$-local ($= K_{(p)*}$-local) spectra. First we easily see that two spectra X and Y in $\underline{Ho}^s$ have equivalent $E(1)_*$-localizations if and only if the pairs $(E(1)_*X, k_X)$ and $(E(1)_*Y, k_Y)$ are isomorphic. Then we prove that for each object $M \in \mathcal{B}(p)_*$ and each element $\kappa \in \mathrm{Ext}^{2,1}_{\mathcal{B}(p)_*}(M, M)$ there exists a spectrum $W \in \underline{Ho}^s$ such that $(E(1)_* W, k_W)$ is isomorphic to (M, κ). Indeed, we prove that this W may be chosen to be a finite CW-spectrum if and only if M is of finite type over $Z_{(p)}$. We arrive at the result that the homotopy types of the $E(1)_*$-local spectra correspond to the isomorphism classes of the pairs (M, κ) with $M \in \mathcal{B}(p)_*$ and $\kappa \in \mathrm{Ext}^{2,1}_{\mathcal{B}(p)_*}(M, M)$. In fact, we obtain much stronger results on the homotopy category $\underline{Ho}^s_{E(1)}$ of $E(1)_*$-local spectra. Using the $E(1)_*$-Adams spectral sequence we algebraically determine the bigraded category obtained from $\underline{Ho}^s_{E(1)}$ by taking Adams filtration quotients, and also determine the full subcategory of $\underline{Ho}^s_{E(1)}$ given by the generalized $E(1)_*$-Moore spectra. In particular, for $M, N \in \mathcal{B}(p)_*$ we obtain canonical isomorphisms

$$[\mathfrak{M}(M), \mathfrak{M}(N)]_n \approx \bigoplus_{s=0}^{2} \mathrm{Ext}^{s,s+n}_{\mathcal{B}(p)_*}(M, N)$$

such that compositions of homotopy classes correspond to Yoneda products. We remark that a spectrum $W \in \underline{Ho}^s_{E(1)}$ is equivalent to a generalized

$E(1)_*$-Moore spectrum if and only if $k_W = 0$. Some examples of spectra with this property are: (i) the $E(1)$-module spectra, (ii) the spectra $W \in \underline{Ho}^s_{E(1)}$ such that $E(1)_* W$ vanishes in all even (or odd) dimensions, and (iii) the spectra XZ/p for arbitrary $X \in \underline{Ho}^s_{E(1)}$. We have not included a proof of the required splitting in case (iii) since we hope to deal with it in a future note.

It would, of course, be desirable to obtain a still more complete algebraization of the homotopy theory of $E(1)_*$-local ($= K_{(p)*}$-local) spectra. In subsequent work we have constructed a certain algebraic homotopy category in which differential injective $\mathcal{B}(p)_*$-objects are used in place of $E(1)_*$-local spectra, and we have obtained results suggesting that our algebraic homotopy category may be equivalent to that of $E(1)_*$-local spectra.

Although we work over an odd prime throughout this paper, we are able to partially extend our results to the prime 2, and we hope to deal with that case in a future paper. In particular, we can extend our algebraic classification of homotopy types to cover the $K_{(2)*}$-local spectral $X \in \underline{Ho}^s$ such that $\eta : KO_{(2)*}X \to KO_{(2)*}X$ is zero or, equivalently, such that the sequence

$$\cdots \longrightarrow K_{(2)*}X \xrightarrow{1+\psi^{-1}} K_{(2)*}X \xrightarrow{1-\psi^{-1}} K_{(2)*}X \xrightarrow{1+\psi^{-1}} \cdots$$

is exact. For such spectra X, the $K_{(2)*}K_{(2)}$-comodules $K_{(2)*}X$ have injective dimension ≤ 2 and the methods of this paper are applicable. To deal with the general case where $K_{(2)*}X$ may have infinite injective dimension, we are developing other methods using $K_{(2)*}X$ together with $KO_{(2)*}X$.

1. The abelian categories $\mathcal{A}(p)$ and $\mathcal{A}(p)_*$. Let p be a fixed odd prime throughout this paper. Our first aim is to construct the abelian category $\mathcal{A}(p)$ whose objects have the formal properties of the homology groups $K_n(X; Z_{(p)}) = K_{(p)n}X$ with their stable Adams operations for $X \in \underline{Ho}^s$ and $n \in Z$ (see Section 2).

Let ψ-*Mod* denote the category of modules over the group ring $Z_{(p)}(Z^*_{(p)})$ where, in general, R^* denotes the multiplicative group of units in a ring R with identity. For $M \in \psi$-*Mod* let $\psi^k : M \approx M$ denote the automorphism given by multiplication by $k \in Z^*_{(p)}$. We construct the finitely generated part of $\mathcal{A}(p)$ as follows. Let $\mathcal{A}(p)_f$ be the full subcategory of ψ-*Mod* given by all $M \in \psi$-*Mod* such that:

(1.1). M is finitely generated over $Z_{(p)}$;

(1.2). for each $m \geq 1$ the action of $Z_{(p)}^*$ on $M/p^m M$ factors through the quotient homomorphism $Z_{(p)}^* \to (Z/p^n)^*$ for sufficiently large n; and

(1.3). the vector space $M \otimes Q$ has a direct sum decomposition $M \otimes Q = \oplus_{i \in Z} W_i$ such that $(\psi^k \otimes 1)w = k^i w$ for each $w \in W_i$, $i \in Z$, and $k \in Z_{(p)}^*$.

LEMMA 1.4. *If M is an object of $\mathfrak{A}(p)_f$ and if H is a ψ-Mod-subobject of M, then H and M/H are in $\mathfrak{A}(p)_f$.*

Proof. To show (1.2) for H, let $m \geq 1$ and choose a sufficiently large s such that $H \cap p^s M \subset p^m H$. Then choose a sufficiently large n such that the action of $Z_{(p)}^*$ on $M/p^s M$ factors through $(Z/p^n)^*$. Since $H/p^m H$ is a subquotient of $M/p^s M$, it follows that the action of $Z_{(p)}^*$ on $H/p^m H$ also factors through $(Z/p^n)^*$. To show (1.3) for H, let $M \otimes Q = \oplus_{i \in Z} W_i$ be given by (1.3) for M. It will suffice to show $H \otimes Q = \oplus_{i \in Z} V_i$ where $V_i = W_i \cap (H \otimes Q)$. For a fixed $k \in Z_{(p)}^*$ with $k \neq \pm 1$, the operator $\psi^k \otimes 1 : M \otimes Q \to M \otimes Q$ is diagonalizable with $W_i = \{w \in M \otimes Q \mid (\psi^k \otimes 1)w = k^i w\}$. Thus $V_i = \{v \in H \otimes Q \mid (\psi^k \otimes 1)v = k^i v\}$. Since the minimal polynomial of $\psi^k \otimes 1 : H \otimes Q \to H \otimes Q$ divides that of $\psi^k \otimes 1 : M \otimes Q \to M \otimes Q$, this restricted operator is also diagonalizable and $H \otimes Q = \oplus_{i \in Z} V_i$ as desired. Thus H is in $\mathfrak{A}(p)_f$, and an easy argument shows that M/H is likewise.

$\mathfrak{A}(p)_f$ is also closed under finite direct sums. Thus $\mathfrak{A}(p)_f$ is an abelian category.

1.5. *The abelian category* $\mathfrak{A}(p)$. For $M \in \psi$-*Mod* and $x \in M$, let $C(x, \psi) \in \psi$-*Mod* denote the ψ-*Mod*-subobject of M generated by x. Let $\mathfrak{A}(p)$ be the full subcategory of ψ-*Mod* given by all $M \in \psi$-*Mod* such that $C(x; \psi)$ is in $\mathfrak{A}(p)_f$ for each $x \in M$. Using the properties of $\mathfrak{A}(p)_f$, one shows that if M is in $\mathfrak{A}(p)$ and if H is a ψ-*Mod*-subobject of M, then H and M/H are in $\mathfrak{A}(p)$. Furthermore $\mathfrak{A}(p)$ is closed under arbitrary direct sums. Hence $\mathfrak{A}(p)$ is an abelian category. One easily checks that the finitely $Z_{(p)}$-generated objects of $\mathfrak{A}(p)$ are the same as the objects of $\mathfrak{A}(p)_f$. Moreover, the p-torsion objects of $\mathfrak{A}(p)$ are the same as the discrete p-torsion abelian groups with continuous $Z_p^\wedge{}^*$-actions, where $Z_p^\wedge{}^* = \lim_n (Z/p^n)^*$ is the topological group of units in the p-adic integers. In particular, for a p-torsion object $M \in \mathfrak{A}(p)$ and $x \in M$ with $p^m x = 0$, the action of $Z_{(p)}^*$ on $C(x; \psi) \in \mathfrak{A}(p)_f$ factors through $(Z/p^n)^*$ for some n because $p^m C(x; \psi) = 0$, and this induces an action by $Z_p^\wedge{}^*$ on $C(x; \psi)$. These actions for the various $x \in M$ combine to give the continuous action by $Z_p^\wedge{}^*$ on M.

For $i \in Z$ and $M \in \psi$-*Mod*, let $T^i M \in \psi$-*Mod* equal M as a $Z_{(p)}$-module but have $\psi^k : T^i M \to T^i M$ equal to $k^i \psi^k : M \to M$ for each $k \in Z_{(p)}^*$. This defines a categorical automorphism $T^i : \psi$-*Mod* $\to \psi$-*Mod*. Let $T = T^1$ and note that T^i is the i^{th} iterate of T.

LEMMA 1.6. *For $i \in Z$, T^i restricts to a categorical automorphism $T^i : \mathfrak{A}(p) \to \mathfrak{A}(p)$.*

Proof. For $M \in \mathfrak{A}(p)_f$ we claim that $T^i M$ is in $\mathfrak{A}(p)_f$. For $m \geq 1$ choose $n \geq m$ such that the action of $Z_{(p)}^*$ on $M/p^m M$ factors through $(Z/p^n)^*$. Then ψ^k equals $\psi^{k+p^n x}$ on $M/p^m M$ for each $k \in Z_{(p)}^*$ and $x \in Z_{(p)}$, and thus $k^i \psi^k$ equals $(k + p^n x)^i \psi^{k+p^n x}$ on $M/p^m M$ since p^n annihilates $M/p^m M$. Thus the action of $Z_{(p)}^*$ on $T^i M/p^m T^i M$ factors through $(Z/p^n)^*$. The required direct sum decomposition of $T^i M \otimes Q$ can be obtained by re-indexing the decomposition of $M \otimes Q$. Thus $T^i M$ is in $\mathfrak{A}(p)_f$, and the lemma follows easily.

1.7. *The abelian category $\mathfrak{A}(p)_*$.* Let $\mathfrak{A}(p)_*$ be the abelian category such that an object $M \in \mathfrak{A}(p)_*$ is a collection of objects $M_n \in \mathfrak{A}(p)$ for $n \in Z$ together with isomorphisms $u : TM_n \approx M_{n+2}$ in $\mathfrak{A}(p)$ for all n, and a morphism $f : M \to N$ in $\mathfrak{A}(p)_*$ is a collection of morphisms $f_n : M_n \to N_n$ in $\mathfrak{A}(p)$ for $n \in Z$ such that $uf_n = f_{n+2}u$ for all n. Note that an object $M \in \mathfrak{A}(p)_*$ is a graded module over the graded algebra $Z_{(p)}[u, u^{-1}]$ with $\deg u = 2$, and has isomorphisms $u^i : T^i M_n \approx M_{n+2i}$ in $\mathfrak{A}(p)$ for all i, $n \in Z$. Thus M is determined (up to a canonical isomorphism) by $M_0, M_1 \in \mathfrak{A}(p)$, and $\mathfrak{A}(p)_*$ is equivalent to the product category $\mathfrak{A}(p) \times \mathfrak{A}(p)$. There is a suspension automorphism $\Sigma^t : \mathfrak{A}(p)_* \to \mathfrak{A}(p)_*$ for $t \in Z$, where $(\Sigma^t M)_n = M_{n-t} \in \mathfrak{A}(p)$ for each $n \in Z$ and where $\Sigma^t M$ has the same u-action as M.

2. The spectrum $K_{(p)}$ and its homology theory.

We now outline for later use some properties of the spectrum $K_{(p)}$ and show that the homology $K_*(X; Z_{(p)}) = K_{(p)*}X$ is in $\mathfrak{A}(p)_*$ for each spectrum X.

Recall that the complex K-theory spectrum $K \in \underline{Ho}^s$ is a commutative ring spectrum with $\pi_* K \approx Z[u, u^{-1}]$ where $u \in \pi_2 K$ is the canonical generator. Thus $K_{(p)} \in \underline{Ho}^s$ is also a commutative ring spectrum with $\pi_* K_{(p)} \approx Z_{(p)}[u, u^{-1}]$. The following lemma (essentially due to Adams-Clarke [6]) shows that all maps $K_{(p)} \wedge \cdots \wedge K_{(p)} \to K_{(p)}$ in $\underline{Ho}^s$ are rationally detectable and are thus detectable by their induced actions $\pi_* K_{(p)} \otimes \cdots \otimes \pi_* K_{(p)} \to \pi_* K_{(p)}$. Let $K_Q \in \underline{Ho}^s$ denote the rationalization of K.

LEMMA 2.1. *The rationalization map*

$$\rho:[K_{(p)}^{\wedge n}, K_{(p)}]_* \to [K_Q^{\wedge n}, K_Q]_*$$

is an injection for $n \geq 1$ (where $X^{\wedge n}$ denotes the smash product of n copies of X).

Proof. By [6], K_*K is a free left π_*K-module. Thus $K^{\wedge n}$ is equivalent to a wedge of copies of K, and $K_*(K^{\wedge n})$ is a free left π_*K-module. By [5], for any K-module spectrum $M \in \underline{Ho}^s$, there is a short exact sequence

$$0 \to \operatorname{Ext}_{\pi_*K}(K_*X, \pi_*M)_{*+1} \to [X, M]_* \xrightarrow{j} \operatorname{Hom}_{\pi_*K}(K_*X, \pi_*M)_* \to 0$$

where j is the obvious map. Thus there is an isomorphism $j:[K^{\wedge n}, M]_* \approx \operatorname{Hom}_{\pi_*K}(K_*(K^{\wedge n}), \pi_*M)_*$, and hence the canonical map $\rho':[K^{\wedge n}, K_{(p)}]_* \to [K^{\wedge n}, K_Q]_*$ is an injection. Since the rationalization map ρ is equivalent to ρ', it is also an injection.

For each $k \in Z_{(p)}^*$ there is an Adams map $\psi^k:K_{(p)} \to K_{(p)}$ in $\underline{Ho}^s$ which carries $u^i \in \pi_{2i}K_{(p)}$ to $k^iu^i \in \pi_{2i}K_{(p)}$ for each $i \in Z$. These may be constructed as composites of classical Adams maps (where k is an integer prime to p) and homotopy inverses of such maps. They satisfy the conditions $\psi^1 = 1$ and $\psi^h \circ \psi^k = \psi^{hk}$ for $h, k \in Z_{(p)}^*$, so that the group $Z_{(p)}^*$ acts on $K_{(p)}$ in $\underline{Ho}^s$. Moreover, $\psi^k:K_{(p)} \to K_{(p)}$ is a ring spectrum equivalence for each $k \in Z_{(p)}^*$. Now for $X \in \underline{Ho}^s$, $K_*(X; Z_{(p)})$ is a graded module over $\pi_*K_{(p)} \approx Z_{(p)}[u, u^{-1}]$, and the Adams maps induce Adams operations $\psi^k:K_*(X; Z_{(p)}) \to K_*(X; Z_{(p)})$ for $k \in Z_{(p)}^*$. Once checks that $\psi^k(u^ix) = k^iu^i\psi^k(x)$ for each $k \in Z_{(p)}^*$, $i \in Z$, and $x \in K_*(X; Z_{(p)})$. Thus, multiplication by u^i induces an isomorphism $u^i:T^iK_n(X; Z_{(p)}) \approx K_{n+2i}(X; Z_{(p)})$ in ψ-*Mod* for each $i, n \in Z$.

PROPOSITION 2.2. *For each $X \in \underline{Ho}^s$, $K_*(X; Z_{(p)})$ is in $\mathfrak{A}(p)_*$.*

Proof. Using the above structure, it suffices to show that $K_n(X; Z_{(p)})$ is in $\mathfrak{A}(p)$ for each $n \in Z$. By Lemma 13.7 of [5], for each $x \in K_nX$ there exists a finite CW-spectrum W and a map $f: W \to X$ such that x is in $f_*(K_nW)$ and K_*W is a free π_*K-module. Since the rationalized spectrum W_Q is equivalent to a wedge of rationalized sphere spectra, $K_n(W_Q; Z_{(p)})$ is clearly in $\mathfrak{A}(p)$. Applying the results of 1.5 to the monomorphism $K_n(W; Z_{(p)}) \to K_n(W_Q; Z_{(p)})$ and to the epimorphism $K_n(W; Z_{(p)}) \to f_*(K_n(W;$

$Z_{(p)})$), one shows that $f_*(K_n(W; Z_{(p)}))$ is in $\mathfrak{A}(p)$. Since each element of $K_n(X; Z_{(p)})$ lies in such an image, $K_n(X; Z_{(p)})$ is also in $\mathfrak{A}(p)$.

For each $X \in \underline{Ho}^s$ and $t \in Z$, there is a natural isomorphism $K_*(\Sigma^t X; Z_{(p)}) \approx \Sigma^t K_*(X; Z_{(p)})$ in $\mathfrak{A}(p)_*$.

3. The abelian categories $\mathfrak{B}(p)$ and $\mathfrak{B}(p)_*$. In our subsequent work it will be convenient to replace the spectrum $K_{(p)}$ by its summand $E(1)$. To permit that, we introduce the abelian category $\mathfrak{B}(p)_*$ which is the $E(1)$-theoretic analogue of $\mathfrak{A}(p)_*$, and we prove that $\mathfrak{B}(p)_*$ is actually equivalent to $\mathfrak{A}(p)_*$. Our first aim is to construct the ungraded abelian category $\mathfrak{B}(p)$ whose objects have the formal properties of the homology groups $E(1)_n X$ with their stable Adams operations for $X \in \underline{Ho}^s$ and $n \in Z$ (see Section 4).

For $n \geq 0$ let Γ^n denote the quotient of $(Z/p^{n+1})^*$ by its subgroup of order $p - 1$, so that Γ^n is a cyclic group of order p^n. We construct the finitely generated part of $\mathfrak{B}(p)$ as follows. Let $\mathfrak{B}(p)_f$ be the full subcategory of ψ-*Mod* given by all $M \in \psi$-*Mod* such that:

(3.1). M is finitely generated over $Z_{(p)}$;

(3.2). for each $m \geq 1$, the action of $Z^*_{(p)}$ on $M/p^m M$ factors through the quotient homomorphism $Z^*_{(p)} \to \Gamma^n$ for sufficiently large n; and

(3.3). the vector space $M \otimes Q$ has a direct sum decomposition $M \otimes Q = \oplus_{j \in Z} W_{j(p-1)}$ such that $(\psi^k \otimes 1)w = k^{j(p-1)}w$ for each $w \in W_{j(p-1)}$, $j \in Z$, and $k \in Z^*_{(p)}$.

As in 1.4, if M is an object $\mathfrak{B}(p)_f$ and if $H \subset M$ is a ψ-*Mod*-subobject of M, then H and M/H are in $\mathfrak{B}(p)_f$. Moreover, $\mathfrak{B}(p)_f$ is also closed under finite direct sums. Thus $\mathfrak{B}(p)_f$ is an abelian category, and it is clearly a full subcategory of $\mathfrak{A}(p)_f$.

3.4. *The abelian category* $\mathfrak{B}(p)$. Let $\mathfrak{B}(p)$ be the full subcategory of ψ-*Mod* given by all $M \in \psi$-*Mod* such that $C(x; \psi)$ is in $\mathfrak{B}(p)_f$ for each $x \in M$. Thus $\mathfrak{B}(p)$ is a full subcategory of $\mathfrak{A}(p)$. As in 1.5, if M is an object of $\mathfrak{B}(p)$ and if H is a ψ-*Mod*-subobject of M, then H and M/H are in $\mathfrak{B}(p)$; and $\mathfrak{B}(p)$ is closed under arbitrary direct sums. Hence $\mathfrak{B}(p)$ *is an abelian category.* One easily checks that the finitely $Z_{(p)}$-generated objects of $\mathfrak{B}(p)$ are the same as the objects of $\mathfrak{B}(p)_f$. Moreover, as in 1.5, the p-torsion objects of $\mathfrak{B}(p)$ are the same as the discrete p-torsion abelian groups with continuous Γ-actions where $\Gamma = \lim_n \Gamma^n$ is the topological quotient group of $Z_p^{\wedge *}$ by its subgroup of order $p - 1$. As noted by Ravenel [11], these torsion objects have been studied by Iwasawa in connection with cyclotomic fields (see [8], [9], [12]).

LEMMA 3.5. *For $j \in Z$ the functor $T^{j(p-1)}: \psi$-Mod $\to \psi$-Mod restricts to a categorical automorphism $T^{j(p-1)}: \mathcal{B}(p) \to \mathcal{B}(p)$.*

Proof. For $M \in \mathcal{B}(p)_f$ and $i = j(p-1)$, we claim that T^iM is in $\mathcal{B}(p)_f$. For $m \geq 1$ choose $n \geq m$ such that the action of $Z_{(p)}^*$ on M/p^mM factors through Γ^{n-1}. Then the action of $Z_{(p)}^*$ on T^iM/p^mT^iM factors through $(Z/p^n)^*$ as in 1.6. If $k \in Z_{(p)}^*$ maps to the identity of Γ^{n-1}, then $k^{p-1} \equiv 1 \bmod p^n$, and thus $k^i\psi^k = \psi^k = 1$ on M/p^mM. Hence the action of $Z_{(p)}^*$ on T^iM/p^mT^iM factors through Γ^{n-1}. The required direct sum decomposition of $(T^iM) \otimes Q$ can be obtained by re-indexing the decomposition of $M \otimes Q$. Thus T^iM is in $\mathcal{B}(p)_f$ and the lemma follows easily.

3.6. *The abelian category $\mathcal{B}(p)_*$.* Let $\mathcal{B}(p)_*$ be the abelian category such that an object $M \in \mathcal{B}(p)_*$ is a collection of objects $M_n \in \mathcal{B}(p)$ for $n \in Z$ together with isomorphisms $v: T^{p-1}M_n \approx M_{n+2p-2}$ in $\mathcal{B}(p)$ for all n, and a morphism $f: M \to N$ in $\mathcal{B}(p)_*$ is a collection of morphisms $f_n: M_n \to N_n$ in $\mathcal{B}(p)$ for $n \in Z$ such that $vf_n = f_{n+2p-2}v$ for all n. Note that an object $M \in \mathcal{B}(p)_*$ is a graded module over the graded algebra $Z_{(p)}[v, v^{-1}]$ with $\deg v = 2p - 2$, and has isomorphisms $v^j: T^{j(p-1)}M_n \approx M_{n+2j(p-1)}$ in $\mathcal{B}(p)$ for all $j, n \in Z$. Thus M is determined (up to a canonical isomorphism) by $M_0, M_1, \ldots, M_{2p-3} \in \mathcal{B}(p)$, and $\mathcal{B}(p)_*$ is equivalent to the $(2p-2)$-fold product category $\mathcal{B}(p) \times \cdots \times \mathcal{B}(p)$. There is a suspension automorphism $\Sigma^t: \mathcal{B}(p)_* \to \mathcal{B}(p)_*$ for $t \in Z$ where $(\Sigma^tM)_n = M_{n-t} \in \mathcal{B}(p)$ for each $n \in Z$ and where Σ^tM has the same v-action as M.

Before proving that $\mathcal{B}(p)_*$ is equivalent to $\mathcal{A}(p)_*$, we must first decompose $\mathcal{A}(p)$ as a product of $p-1$ subcategories which are equivalent to $\mathcal{B}(p)$. For $i \in Z$ let $T^i\mathcal{B}(p)$ be the full subcategory of $\mathcal{A}(p)$ given by all T^iN for $N \in \mathcal{B}(p)$ (or equivalently by all $M \in \mathcal{A}(p)$ with $T^{-i}M$ in $\mathcal{B}(p)$). Note that $T^i\mathcal{B}(p)$ is categorically equivalent to $\mathcal{B}(p)$ for each $i \in Z$, and that $T^i\mathcal{B}(p)$ equals $T^j\mathcal{B}(p)$ for $i \equiv j \bmod p-1$. For $M \in \mathcal{A}(p)$ and $i \in Z$, let $M^{[i]} \subset M$ be the natural subobject

$$M^{[i]} = \{x \in M \mid C(x; \psi) \in T^i\mathcal{B}(p)\}.$$

Clearly $M^{[i]}$ is in $T^i\mathcal{B}(p)$ and contains all subobjects of M in $T^i\mathcal{B}(p)$. Also $M^{[i]} = M^{[j]}$ for $i \equiv j \bmod p-1$.

PROPOSITION 3.7. *Each $M \in \mathcal{A}(p)$ is the direct sum of its natural subobjects $M^{[i]} \in T^i\mathcal{B}(p)$ for $i = 0, 1, \ldots, p-2$. Thus $\mathcal{A}(p)$ is the product of its subcategories $T^i\mathcal{B}(p)$ for $i = 0, 1, \ldots, p-2$. Furthermore, $T^i(M^{[j]}) = (T^iM)^{[i+j]}$ for $M \in \mathcal{A}(p)$ and $i, j \in Z$.*

This will be proved in 3.10, and it implies

3.8. *The equivalence of* $\mathcal{B}(p)_*$ *and* $\mathcal{A}(p)_*$. Using the embedding $\omega: Z_{(p)}[v, v^{-1}] \to Z_{(p)}[u, u^{-1}]$ of graded algebras with $\omega(v) = u^{p-1}$, define a functor $L: \mathcal{B}(p)_* \to \mathcal{A}(p)_*$ by $L(N) = Z_{(p)}[u, u^{-1}] \otimes_{Z_{(p)}[v,v^{-1}]} N$ for $N \in \mathcal{B}(p)_*$ where $\psi^k(u^m \otimes x) = k^m u^m \otimes \psi^k x$ for $k \in Z^*_{(p)}$, $m \in Z$, and $x \in N$. Thus $(LN)_n \approx \bigoplus_{i=0}^{p-2} T^i N_{n-2i}$ in $\mathcal{A}(p)$ for each $n \in Z$. Also define a functor $R: \mathcal{A}(p)_* \to \mathcal{B}(p)_*$ by $R(M) = M^{[0]}$ for $m \in \mathcal{A}(p)_*$ where $vx = u^{p-1}x$ for each $x \in M^{[0]}$. Now 3.7 implies that the functors $L: \mathcal{B}(p)_* \to \mathcal{A}(p)_*$ and $R: \mathcal{A}(p)_* \to \mathcal{B}(p)_*$ are equivalences with L left adjoint to R. These equivalences respect suspensions.

For the proof of 3.7 we need a lemma involving the cyclic group $J = \{j \in Z_p^{\wedge *} \mid j^{p-1} = 1\}$ of order $p - 1$. Consider the p-adic group ring $Z_p^\wedge J$ and let $[j] \in Z_p^\wedge J$ denote the element corresponding to $j \in J$.

LEMMA 3.9. *In* $Z_p^\wedge J$ *there exist elements* $e_0, e_1, \ldots, e_{p-2}$ *such that*:

(i) $e_i e_i = e_i$ *and* $e_i e_k = 0$ *for each* $i \neq k$.
(ii) $e_0 + e_1 + \cdots + e_{p-2} = 1$
(iii) $[j]e_i = j^i e_i$ *for each* $j \in J$ *and each* i.

Proof. Let ξ be a generator of J. In the polynomial algebra $Z_p^\wedge[x]$, note that

$$x^{p-1} - 1 = (x - \xi^0)(x - \xi^1) \cdots (x - \xi^{p-2})$$

and let $\delta_i \in Z_p^\wedge[x]$ denote $(x^{p-1} - 1)/(x - \xi^i)$ for $0 \leq i \leq p - 2$. Using the ring homomorphism $\varphi: Z_p^\wedge[x] \to Z_p^\wedge J$ with $\varphi(x) = [\xi]$, let $d_i = \varphi(\delta_i)$ in $Z_p^\wedge J$ for each i. Then $([\xi] - \xi^i)d_i = 0$ and $d_i d_k = 0$ for each $i \neq k$ since $\ker \varphi$ is the ideal generated by $x^{p-1} - 1$. Using the quotient map $Z_p^\wedge \to F_p$ to the prime field F_p, let $\bar{d}_i \in F_p J$ be the image of d_i for each i. Then $\bar{d}_0, \ldots, \bar{d}_{p-2}$ form a F_p-basis for $F_p J$ because they are eigenvectors of $[\xi]: F_p J \to F_p J$ with distinct eigenvalues. Consequently, $d_0, \ldots, d_{p-2}$ form a $Z_p^\wedge$-basis for the free $Z_p^\wedge$-module $Z_p^\wedge J$. Thus there exist elements $a_0, \ldots, a_{p-2} \in Z_p^\wedge$ such that $a_0 d_0 + \cdots + a_{p-2} d_{p-2} = 1$ in $Z_p^\wedge J$, and we let $e_i = a_i d_i$ for each i. Clearly $e_i e_k = 0$ for each $i \neq k$, $e_0 + \cdots + e_{p-2} = 1$, and $[\xi]e_i = \xi^i e_i$ for each i. The other required properties follow trivially.

3.10. *Proof of* 3.7. We first show $M = \bigoplus_{i=0}^{p-2} M^{[i]}$ for $M \in \mathcal{A}(p)_f$. For $m \geq 1$, the $Z_p^{\wedge *}$-action on $M/p^m M$ in 1.5 restricts to a J-action, so $M/p^m M$ is a $Z_p^\wedge J$-module. Thus there is a decomposition $M/p^m M = \bigotimes_{i=0}^{p-2} e_i(M/p^m M)$ by 3.9. We form the arithmetic square

$$\begin{array}{ccc} M & \longrightarrow & M_p^\wedge \\ \downarrow & & \downarrow \\ M \otimes Q & \xrightarrow{c \otimes 1} & M_p^\wedge \otimes Q \end{array}$$

which is a pull-back diagram in ψ-*Mod* involving the p-adic completion $M_p^\wedge = \lim_m M/p^m M$, and we now have the following decompositions:

$$M_p^\wedge = \bigoplus_{i=0}^{p-2} P_i M_p^\wedge \quad \text{where} \quad P_i M_p^\wedge = \lim_m e_i(M/p^m M),$$

$$M_p^\wedge \otimes Q = \bigoplus_{i=0}^{p-2} P_i(M_p^\wedge \otimes Q) \quad \text{where} \quad P_i(M_p^\wedge \otimes Q) = P_i M_p^\wedge \otimes Q, \quad \text{and}$$

$$M \otimes Q = \bigoplus_{i=0}^{p-2} P_i(M \otimes Q) \quad \text{where} \quad P_i(M \otimes Q) = \bigoplus_{j \in Z} W_{i+j(p-1)}$$

using the eigenspace decomposition $M \otimes Q = \oplus_{i \in Z} W_i$ of Section 1. We claim that $c \otimes 1 : M \otimes Q \to M_p^\wedge \otimes Q$ respects the P_i-decompositions. To see this, choose a finitely generated free $Z_{(p)}$-module $F \subset M \otimes Q$ such that F contains the image of $M \to M \otimes Q$ and has a decomposition $F = \oplus_{i \in Z} F_i$ with $F_i \subset W_i$ for each $i \in Z$. Now $c \otimes 1 : F \otimes Q \to F_p^\wedge \otimes Q$ clearly respects the P_i-decompositions, and the above claim follows by a naturality argument using the map $M \to F$ in $\mathcal{A}(p)_f$. Let $M = \oplus_{i=0}^{p-2} P_i M$ be the decomposition induced via the arithmetic square. It is straightforward to check that $P_i M$ is in $T^i \mathcal{B}(p)$ and $P_i(M^{[i]}) = M^{[i]}$ for each i. Consequently, $P_i M = M^{[i]}$ and $M = \oplus_{i=0}^{p-2} M^{[i]}$. The proposition now follows easily.

4. The spectrum $E(1)$ and its homology theory. It has been shown by Adams [2] and Anderson-Meiselman (unpublished) that the spectrum $K_{(p)}$ splits into a wedge $\vee_{i=0}^{p-2} \Sigma^{2i} E(1)$ involving a certain spectrum $E(1) \in \underline{Ho}^s$. We now recover that result and develop basic properties of the spectrum $E(1)$ corresponding to those of $K_{(p)}$. We show that $E(1)_* X$ is in the abelian category $\mathcal{B}(p)_*$ for each $X \in \underline{Ho}^s$ and explain how $E(1)_* X$ and $K_{(p)*} X$ determine each other.

4.1. *The spectrum* $E(1)$. By 3.7 the homology theory $K_{(p)*}$ on $\underline{Ho}^s$ contains a homology subtheory $(K_{(p)*})^{[0]}$. Thus by the Brown representability theorem for homology theories (see 4.3), there exists a spectrum $E(1)$

and map $\omega: E(1) \to K_{(p)}$ in $\underline{Ho}^s$ such that the induced map $E(1)_*X \to K_{(p)*}X$ is an injection onto $(K_{(p)*}X)^{[0]}$ for each $X \in \underline{Ho}^s$. We choose such an $E(1)$ and ω. Using $X = S$ we see that $\omega_*: \pi_i E(1) \approx \pi_i K_{(p)} \approx Z_{(p)}$ for $i \equiv 0 \bmod 2p - 2$ and $\pi_i E(1) = 0$ otherwise. Thus there is an equivalence $\vee_{i=0}^{p-2} \Sigma^{2i} E(1) \xrightarrow{\simeq} K_{(p)}$ given by $u^i \omega$ on $\Sigma^{2i}E(1)$. Now 2.1 implies that the rationalization map $[E(1)^{\wedge n}, E(1)]_* \to [E(1)_Q^{\wedge n}, E(1)_Q]_*$ is an injection for $n \geq 1$, and consequently $[E(1)^{\wedge n}, E(1)]_i = 0$ unless $i \equiv 0 \bmod 2p - 2$. Hence $E(1)$ inherits structures corresponding to: the multiplication $\mu: K_{(p)} \wedge K_{(p)} \to K_{(p)}$, the unit $\alpha: S \to K_{(p)}$, and the Adams maps $\psi^k: K_{(p)} \to K_{(p)}$ for $k \in Z_{(p)}^*$. Specifically, there exist unique maps $\mu: E(1) \wedge E(1) \to E(1)$, $\alpha: S \to E(1)$, and $\psi^k: E(1) \to E(1)$ in $\underline{Ho}^s$ for $k \in Z_{(p)}^*$ such that

$$\begin{tikzcd} E(1) \wedge E(1) \arrow[r, "\mu"] \arrow[d, "\omega\wedge\omega"] & E(1) \arrow[d, "\omega"] \\ K_{(p)} \wedge K_{(p)} \arrow[r, "\mu"] & K_{(p)} \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r, "\alpha"] \arrow[d, "1"] & E(1) \arrow[d, "\omega"] \\ S \arrow[r, "\alpha"] & K_{(p)} \end{tikzcd} \qquad \begin{tikzcd} E(1) \arrow[r, "\psi^k"] \arrow[d, "\omega"] & E(1) \arrow[d, "\omega"] \\ K_{(p)} \arrow[r, "\psi^k"] & K_{(p)} \end{tikzcd}$$

commute in $\underline{Ho}^s$. With these maps, $E(1)$ is a commutative ring spectrum, and $\psi^k: E(1) \to E(1)$ is a ring spectrum equivalence with $\psi^1 = 1$ and $\psi^h \circ \psi^k = \psi^{hk}$ for $h, k \in Z_{(p)}^*$ because of the corresponding properties for $K_{(p)}$. Let $v \in \pi_{2p-2}E(1)$ denote the element such that $\omega_* v = u^{p-1}$ in $\pi_{2p-2}K_{(p)}$. Then clearly $\pi_* E(1) = Z_{(p)}[v, v^{-1}]$, and $\psi^k: E(1) \to E(1)$ carries v^j to $k^{j(p-1)}v^j$ in $\pi_{2j(p-1)}E(1)$ for each $j \in Z$ and $k \in Z_{(p)}^*$.

4.2. *The homology theory* $E(1)_*$. For $X \in \underline{Ho}^s$, $E(1)_*X$ is a graded module over $\pi_*E(1) = Z_{(p)}[v, v^{-1}]$ and the Adams maps $\psi^k: E(1) \to E(1)$ induce Adams operations $\psi^k: E(1)_*X \to E(1)_*X$ for $k \in Z_{(p)}^*$. One checks that $\psi^k(v^j x) = k^{j(p-1)} v^j \psi^k(x)$ for each $k \in Z_{(p)}^*$, $j \in Z$, and $x \in E(1)_*X$. Thus, multiplication by v^j induces an isomorphism $v^j: T^{j(p-1)}E(1)_n X \approx E(1)_{n+2j(p-1)}X$ in ψ-*Mod* for each j, $n \in Z$. Since $\omega_*: E(1)_*X \approx K_*(X; Z_{(p)})^{[0]}$ is an isomorphism respecting the Adams operations, it now follows *that* $E(1)_*X$ *is in* $\mathcal{B}(p)_*$ *for each* $X \in \underline{Ho}^s$. *Moreover, the natural isomorphism* $w_*: E(1)_*X \approx K_*(X; Z_{(p)})^{[0]}$ *in* $\mathcal{B}(p)_*$ *is adjoint to the natural isomorphism* $\pi_*K_{(p)} \otimes_{\pi_*E(1)} E(1)_*X \approx K_*(X; Z_{(p)})$ *in* $\mathcal{A}(p)_*$ by 3.8. Also, for $t \in Z$, there is a natural isomorphism $E(1)_*(\Sigma^t X) \approx \Sigma^t(E(1)_*X)$ in $\mathcal{B}(p)_*$.

In our construction of $E(1)$, we used the following variant of the Brown representability theorem. It is due to Adams and is implicitly contained in [3]. Let $\underline{Ab}$ denote the category of abelian groups. A functor

$h:\underline{Ho}^s \to \underline{Ab}$ is called *half-exact* if it carries each cofibre sequence $X \to Y \to Z$ to an exact sequence $h(X) \to h(Y) \to h(Z)$, and h is called *convergent* if $h(X)$ is the colimit of $\{h(X_\alpha)\}$ for each $X \in \underline{Ho}^s$ where X_α runs over the finite CW-subspectra of X. By a *homology theory* h_* on $\underline{Ho}^s$ we mean a collection of convergent half-exact functors $h_n:\underline{Ho}^s \to \underline{Ab}$ and natural equivalences $h_n \approx h_{n+1}\Sigma$ for $n \in Z$.

THEOREM 4.3. (Adams). *Each homology theory h_* on $\underline{Ho}^s$ is naturally to A_* for some $A \in \underline{Ho}^s$. Each natural transformation of homology theories $A_* \to B_*$ for A, $B \in \underline{Ho}^s$ is induced by some (not necessarily unique) map $A \to B$ in $\underline{Ho}^s$.*

Proof. Let $\underline{Ho}^s_f$ denote the homotopy category of finite CW-spectra. Adams' arguments in [3] apply in the context of CW-spectra to prove that each contravariant half-exact functor $H:\underline{Ho}^s_f \to \underline{Ab}$ is naturally equivalent to $[-,A]:\underline{Ho}^s_f \to \underline{Ab}$ for some $A \in \underline{Ho}^s$, and that each natural transformation from $[-,A]:\underline{Ho}^s_f \to \underline{Ab}$ to $[-,B]:\underline{Ho}^s_f \to \underline{Ab}$ is induced by some (not necessarily unique) map $A \to B$ in $\underline{Ho}^s$. Given a homology theory h_* on $\underline{Ho}^s$, let $H:\underline{Ho}^s_f \to \underline{Ab}$ be the contravariant functor with $H(X) = h_0(DX)$ for $X \in \underline{Ho}^s_f$ where $D:\underline{Ho}^s_f \to \underline{Ho}^s_f$ is the Spanier-Whitehead duality functor. By the above result, H is equivalent to $[-,A]:\underline{Ho}^s_f \to \underline{Ab}$ for some $A \in \underline{Ho}^s$, and one easily deduces that h_* is equivalent to A_*. The second part of the theorem follows similarly.

5. Simplified constructions of $\mathcal{B}(p)$ and $\mathcal{B}(p)_*$. Recall that Γ^n denotes the quotient of $(Z/p^{n+1})^*$ by its subgroup of order $p - 1$, so that Γ^n is a cyclic group of order p^n. Let r be a fixed integer generating Γ^1, and therefore generating Γ^n for each $n \geq 1$. We now show that the operations ψ^k on an object $M \in \mathcal{B}(p)_*$ are all canonically determined by the single operation ψ^r, and we show that $\mathcal{B}(p)_*$ can be identified with a certain category $\mathcal{B}(p)^r_*$ involving only the operation ψ^r. Similar results have been obtained for torsion Γ-modules by Serre [12] and for torsion $E(1)_*E(1)$-comodules by Ravenel [11].

We begin by constructing a category $\mathcal{B}(p)^r$ which can be identified with $\mathcal{B}(p)$. Let ψ^r-*Mod* denote the category whose objects are $Z_{(p)}$-modules equipped with an endomorphism denoted by ψ^r, and whose morphisms are $Z_{(p)}$-homomorphisms commuting with ψ^r. Let $\mathcal{B}(p)^r$ be the full subcategory of ψ^r-*Mod* given by all $M \in \psi^r$-*Mod* such that:

(5.1). for each p-torsion element $x \in M$, there exists $u \geq 0$ such that $(\psi^r)^{p^u}x = x$, and

(5.2). the operator $\psi^r \otimes 1$ on $M \otimes Q$ is diagonalizable with eigenvalues all of the form $r^{j(p-1)}$ for $j \in Z$ (or equivalently $M \otimes Q = \oplus_{j \in Z} W_{j(p-1)}$ where $W_{j(p-1)} = \{x \in M \otimes Q \mid \psi^r x = r^{j(p-1)}x\}$.)

Letting $\tilde{\psi}^r = \psi^r - 1$ it is often convenient to replace condition (5.1) by:

(5.3). For each p-torsion element $x \in M$, there exists $h \geq 1$ such that $(\tilde{\psi}^r)^h x = 0$.

LEMMA 5.4. *For an object $M \in \psi^r$-Mod,* (5.1) *is equivalent to* (5.3).

The proof is in 5.11. In view of 5.4 a torsion object in $\mathcal{B}(p)^r$ is merely a p-torsion abelian group equipped with a locally nilpotent endomorphism $\tilde{\psi}^r$. Clearly $\mathcal{B}(p)^r$ is closed under arbitrary direct sums in ψ^r-*Mod*, and $\mathrm{B}(p)^r$ *is an abelian category* since:

LEMMA 5.5. *If $M \in \mathcal{B}(p)^r$ and if H is a ψ^r-Mod-subobject of M, then H and M/H are in $\mathcal{B}(p)^r$.*

The proof is in 5.12.

For $i \in Z$ and $M \in \psi^r$-*Mod*, let $T^i M \in \psi^r$-*Mod* equal M as a $Z_{(p)}$-module but have $\psi^r : T^i M \to T^i M$ equal to $r^i \psi^r : M \to M$. This defines a categorical automorphism $T^i : \psi^r\text{-}Mod \to \psi^r\text{-}Mod$. It is straightforward to show that $T^{j(p-1)}$ restricts to a categorical automorphism $T^{j(p-1)} : \mathcal{B}(p)^r \to \mathcal{B}(p)^r$ for $j \in Z$.

5.6. *The abelian category* $\mathcal{B}(p)^r_*$. Let $\mathcal{B}(p)^r_*$ be the abelian category such that an object $M \in \mathcal{B}(p)^r_*$ is a collection of objects $M_n \in \mathcal{B}(p)^r$ for $n \in Z$ together with isomorphisms $v : T^{p-1} M_n \approx M_{n+2p-2}$ in $\mathcal{B}(p)^r$ for all n, and a morphism $f : M \to N$ in $\mathcal{B}(p)^r_*$ is a collection of morphisms $f_n : M_n \to N_n$ in $\mathcal{B}(p)^r$ for $n \in Z$ such that $vf_n = f_{n+2p-2}v$ for all n. For $t \in Z$, there is a suspension automorphism $\Sigma^t : \mathcal{B}(p)^r_* \to \mathcal{B}(p)^r_*$ defined as in 3.6.

We now show that the categories $\mathcal{B}(p)^r_*$ and $\mathcal{B}(p)^r$ can respectively be identified with $\mathcal{B}(p)_*$ and $\mathcal{B}(p)$. First note that there is a forgetful functor $\varphi : \mathcal{B}(p) \to \mathcal{B}(p)^r$. In particular, if $x \in M \in \mathcal{B}(p)$ with $p^m x = 0$, then $C(x; \psi)$ is in $\mathcal{B}(p)_f$ with $p^m C(x; \psi) = 0$, and thus the action of $Z^*_{(p)}$ on $C(x; \psi)$ factors through Γ^n for some n. Since Γ^n is of order p^n, it follows that $(\psi^r)^{p^n} x = x$. The functor $\varphi : \mathcal{B}(p) \to \mathcal{B}(p)^r$ clearly prolongs to a forgetful functor $\varphi : \mathcal{B}(p)_* \to \mathcal{B}(p)^r_*$.

PROPOSITION 5.7. *The forgetful functor $\varphi : \mathcal{B}(p)_* \to \mathcal{B}(p)^r_*$ (resp. $\varphi : \mathcal{B}(p) \to \mathcal{B}(p)^r$) is an isomorphism of categories. In more detail, the structure of each object in $\mathcal{B}(p)^r_*$ (resp. $\mathcal{B}(p)^r$) extends uniquely to a* *structure in* $\mathcal{B}(p)_*$ *(resp.* $\mathcal{B}(p)$*), and these extensions provide a functor which is inverse to* φ*. These functors respect suspensions.*

The proof is in 5.13, and we now give some preparatory lemmas. We say that an object $M \in \psi^r\text{-}Mod$ has *Property A* if M is finitely generated over $Z_{(p)}$ and if for each integer $m \geq 1$ there exists $u \geq 0$ such that $(\psi^r)^{p^u}$ acts as the identity on $M/p^m M$.

LEMMA 5.8. *If* $M \in \psi^r\text{-}Mod$ *has Property A and if* H *is a* $\psi^r\text{-}Mod$*-subobject of* M*, then* H *and* M/H *have Property A.*

The proof is similar to that of 1.4.

LEMMA 5.9. *If* $M \in \mathcal{B}(p)^r$ *is finitely generated over* $Z_{(p)}$*, then* M *has Property A.*

Proof. Let $\tilde{M}$ be the p-torsion subgroup of M, and let $\bar{M} = M/\tilde{M}$. Using the diagonalizability property of $\psi^r \otimes 1$ on $M \otimes Q$, one constructs a free $Z_{(p)}$-module F generated by a set of eigenvalues in $M \otimes Q$ and such that $\bar{M} \subset F \subset M \otimes Q$. One easily shows that F has Property A, and thus $\bar{M}$ also does by 5.8. Now for a given $m \geq 1$ consider the short exact sequence

$$0 \to \tilde{M}/p^m\tilde{M} \to M/p^m M \to \bar{M}/p^m\bar{M} \to 0$$

and choose $v \geq 0$ such that $(\psi^r)^{p^v}$ acts as the identity on both $\tilde{M}/p^m\tilde{M}$ and $\bar{M}/p^m\bar{M}$. Let G be the group of all automorphisms of $M/p^m M$ which act as the identity on both $\tilde{M}/p^m\tilde{M}$ and $\bar{M}/p^m\bar{M}$. Then G has order p^w for some $w \geq 0$ since it is isomorphic to the additive group of homomorphisms from $\bar{M}/p^m\bar{M}$ to $\tilde{M}/p^m\tilde{M}$. Consequently $(\psi^r)^{p^{v+w}}$ acts as the identity on $M/p^m M$.

For $M \in \psi^r\text{-}Mod$ and $x \in M$, let $C(x; \psi^r) \in \psi^r\text{-}Mod$ denote the $\psi^r\text{-}Mod$ subobject of M generated by x (i.e., $C(x; \psi^r)$ is generated over $Z_{(p)}$ by the elements $(\psi^r)^i x$ for all $i \geq 0$).

LEMMA 5.10. *If* $M \in \mathcal{B}(p)^r$ *and* $x \in M$*, then* $C(x; \psi^r)$ *is finitely generated over* $Z_{(p)}$.

Proof. Let $\tilde{M}$ be the p-torsion subgroup of M, let $\bar{M} = M/\tilde{M}$, and let $\bar{x} \in \bar{M}$ be the image of x. Using the diagonalizability property of $\psi^r \otimes 1$ on $M \otimes Q$ one shows that $C(\bar{x}; \psi^r)$ is finitely generated over $Z_{(p)}$. Thus there exist $q \geq 0$ and $a_0, \ldots, a_q \in Z_{(p)}$ such that $(\psi^r)^{q+1}\bar{x} = \Sigma_{i=0}^q a_i(\psi^r)^i\bar{x}$. Consequently $(\psi^r)^{q+1}x = \tilde{x} + \Sigma_{i=0}^q a_i(\psi^r)^i x$ for some $\tilde{x} \in$

$\tilde{M}$. Since the set $R = \{(\psi^r)^i\tilde{x} \mid i \geq 0\}$ is finite, it follows that $C(x; \psi^r)$ is finitely generated over $Z_{(p)}$ by $R \cup S$ where $S = \{(\psi^r)^i x \mid 0 \leq i \leq q\}$.

5.11. *Proof of* 5.4. First suppose (5.1) and let $x \in M$ be a p-torsion element with $(\psi^r)^{p^u}x = x$. Then $C(x; \psi^r)$ is a finite p-group and has a finite filtration $\{F_i\}_{i \geq 0}$ with $F_i = p^i C(x; \psi^r)$. Then $(\psi^r - 1)^{p^u} = (\psi^r)^{p^u} - 1 = 0$ on each of the Z/p-modules F_i/F_{i+1}. An easy extension argument now shows that $\psi^r - 1$ acts nilpotently on $C(x; \psi^r)$, and this implies (5.3). The reverse implication may be proved similarly, using an extension argument like that in the proof of 5.9.

5.12. *Proof of* 5.5. The diagonalizability property (5.2) for $M \otimes Q$ holds also for $H \otimes Q$ and $(M/H) \otimes Q$ by elementary linear algebra. Thus H is clearly in $\mathcal{B}(p)^r$, and it remains to show the periodicity property (5.1) of ψ^r on an element $\bar{x} \in M/H$ with $p^m\bar{x} = 0$ for some $m \geq 1$. Choose $x \in M$ representing $\bar{x}$. Then the ψ^r-*Mod*-subobject $C(x; \psi^r) \subset M$ is in $\mathcal{B}(p)^r$ and is finitely generated over $Z_{(p)}$ by 5.10. Hence $C(x; \psi^r)$ has Property A by 5.9, and its image $C(\bar{x}; \psi^r)$ must also have Property A. Thus, since $p^m C(\bar{x}; \psi^r) = 0$, ψ^r has the desired periodicity property on $\bar{x}$.

5.13. *Proof of* 5.7. We shall first impose a canonical $\mathcal{B}(p)$-structure on an object $M \in \mathcal{B}(p)^r$ which is finitely generated over $Z_{(p)}$. We use the arithmetic square

$$\begin{array}{ccc} M & \xrightarrow{c} & M_p^\wedge \\ \downarrow & & \downarrow \\ M \otimes Q & \xrightarrow{c \otimes 1} & M_p^\wedge \otimes Q \end{array}$$

which is a pull-back diagram in ψ^r-*Mod* involving the p-adic completion $M_p^\wedge = \lim_m M/p^m M$. Clearly $M \otimes Q$ has a unique ψ-*Mod*-structure such that $\psi^k w = k^{j(p-1)}w$ for each $k \in Z_{(p)}^*$ and $w \in W_{j(p-1)}$ where $M \otimes Q = \bigoplus_{j \in Z} W_{j(p-1)}$ is the eigenspace decomposition given by (5.2). Also, for $m \geq 1$, $M/p^m M$ has a unique ψ-*Mod*-structure such that $\psi^{r^i} = (\psi^r)^i$ on $M/p^m M$ for each $i \geq 0$ and such that the action of $Z_{(p)}^*$ on $M/p^m M$ factors through Γ^n for some n. This follows because $(\psi^r)^{p^n}$ acts as the identity on $M/p^m M$ for some $n \geq 0$ by 5.9 and because r generates the cyclic group Γ^n of order p^n. The ψ-*Mod*-structures of $M/p^m M$ for $m \geq 1$ determine ψ-*Mod*-structures of $M_p^\wedge$ and $M_p^\wedge \otimes Q$. To show that $c \otimes 1 : M \otimes Q \to M_p^\wedge \otimes Q$ is in ψ-*Mod*, let $\bar{M} \subset F \subset M \otimes Q$ be as in the proof of 5.9. Since $F \in$

$\mathcal{B}(p)^r$ is a free $Z_{(p)}$-module on a finite set of eigenvectors of $\psi^r \otimes 1: M \otimes Q \to M \otimes Q$, one easily proves that $c \otimes 1: F \otimes Q \to F_p^\wedge \otimes Q$ is in ψ-*Mod*. Since the vertical maps in the commutative diagram

$$\begin{array}{ccc} M \otimes Q & \xrightarrow{c \otimes 1} & M_p^\wedge \otimes Q \\ \downarrow & & \downarrow \\ F \otimes Q & \xrightarrow{c \otimes 1} & F_p^\wedge \otimes Q \end{array}$$

are monomorphisms in ψ-*Mod*, $c \otimes 1: M \otimes Q \to M_p^\wedge \otimes Q$ is also in ψ-*Mod*. Now let $\theta(M) \in \psi$-*Mod* denote M equipped with the ψ-*Mod* structure determined by the arithmetic square. One checks that $\theta(M)$ is in $\mathcal{B}(p)$, that this is the unique $\mathcal{B}(p)$-structure on M extending its $\mathcal{B}(p)^r$-structure, and that θ is functorial. Next, for an arbitrary object $N \in \mathcal{B}(p)^r$, we let $\theta(N)$ denote N equipped with the ψ-*Mod*-structure which restricts to $\theta C(x; \psi^r)$ for each $x \in N$. One checks that $\theta(N)$ is in $\mathcal{B}(p)$, that this is the unique $\mathcal{B}(p)$-structure on N extending its $\mathcal{B}(p)^r$-structure, and that θ is still functorial. Now 5.7 follows easily.

6. Universal objects in $\mathcal{B}(p)$ and $\mathcal{B}(p)_*$. Let $\pi_*E(1)$-*Mod* denote the category of graded modules over $\pi_*E(1) \approx Z_{(p)}[v, v^{-1}]$. We shall construct a functor $\mathcal{U}: \pi_*E(1)$-*Mod* $\to \mathcal{B}(p)_*$ which is right adjoint to the forgetful functor, and we shall obtain a canonical isomorphism $E(1)_*Y \approx \mathcal{U}(\pi_*Y)$ in $\mathcal{B}(p)_*$ for each $E(1)$-module spectrum Y. We begin by constructing the corresponding ungraded functor $\mathcal{U}: Z_{(p)}$-*Mod* $\to \mathcal{B}(p)$ on the category of $Z_{(p)}$-modules. Whenever convenient, we identify $\mathcal{B}(p)$ (resp. $\mathcal{B}(p)_*$) with $\mathcal{B}(p)^r$ (resp. $\mathcal{B}(p)_*^r$) using 5.7 where r is a fixed integer generating Γ^1.

Definition 6.1. A *universal* $\mathcal{B}(p)$-*object* over a $Z_{(p)}$-module G is an object $\mathcal{U}(G) \in \mathcal{B}(p)$ together with a $Z_{(p)}$-homomorphism $e: \mathcal{U}(G) \to G$ such that for each $X \in \mathcal{B}(p)$ and each $Z_{(p)}$- homomorphism $f: X \to G$ there exists a unique map $\bar{f}: X \to \mathcal{U}(G)$ in $\mathcal{B}(p)$ with $e\bar{f} = f$.

Such a universal $\mathcal{B}(p)$-object over G is clearly unique up to a canonical isomorphism. We now construct it in two important cases and prove that it always exists.

For a p-torsion abelian group G, let $\mathcal{U}(G) = \oplus_{n=1}^\infty G_n$ where each G_n is a copy of G, and let $\bar{\psi}^r: \mathcal{U}(G) \to \mathcal{U}(G)$ send $(g_1, g_2, \ldots)$ to $(g_2, g_3, \ldots)$. By 5.4 and 5.7, this determines a $\mathcal{B}(p)$-structure for $\mathcal{U}(G)$ with $\psi^r =$

$\tilde{\psi}^r + 1$. We claim that $e:\mathcal{U}(G) \to G$ is a universal $\mathcal{B}(p)$-object over G where e sends $(g_1, g_2, \ldots)$ to g_1. For each $X \in \mathcal{B}(p)$ and each $Z_{(p)}$-homomorphism $f:X \to G$, a lifting $\bar{f}:X \to \mathcal{U}(G)$ in $\mathcal{B}(p)$ must send $x \in X$ to $(fx, f\tilde{\psi}^r x, f(\tilde{\psi}^r)^2 x, \ldots)$, and it suffices to show $f(\tilde{\psi}^r)^h x = 0$ for sufficiently large h. Since $C(x; \psi^r)$ is finitely generated over $Z_{(p)}$, we may choose $m \geq 1$ such that $p^m f C(x; \psi^r) = 0$. Then for sufficiently large h, $(\tilde{\psi}^r)^h$ equals 0 on $C(x; \psi^r)/p^m C(x; \psi^r)$ and therefore $f(\tilde{\psi}^r)^h x = 0$.

For a rational (i.e., uniquely divisible) abelian group G, let $G = \bigoplus_{j \in Z} G^{j(p-1)}$ where each $G^{j(p-1)}$ is a copy of G, and let $\psi^k:\mathcal{U}(G) \to \mathcal{U}(G)$ be given by $k^{j(p-1)}:G^{j(p-1)} \to G^{j(p-1)}$ for each $j \in Z$ and $k \in Z_{(p)}^*$. Then $e:\mathcal{U}(G) \to G$ is a universal $\mathcal{B}(p)$-object over G where e is given by $1:G^{j(p-1)} \to G$ for each $j \in Z$. For each $X \in \mathcal{B}(p)$ and each $Z_{(p)}$-homomorphism $f:X \to G$, the unique lifting $\bar{f}:X \to \mathcal{U}(G)$ in $\mathcal{B}(p)$ is obtained by factoring f through $X \otimes Q$ and using the eigenspace decomposition of $X \otimes Q$.

PROPOSITION 6.2. *Over each $Z_{(p)}$-module G, there exists a universal $\mathcal{B}(p)$-object $e:\mathcal{U}(G) \to G$.*

Proof. If G is divisible, then $G = I \oplus J$ where I is rational and J is p-torsion, and we let $e:\mathcal{U}(G) \to G$ be the direct sum of the universal $\mathcal{B}(p)$-objects $e:\mathcal{U}(I) \to I$ and $e:\mathcal{U}(J) \to J$ constructed above. In general, we construct a short exact sequence $0 \to G \to D^0 \to D^1 \to 0$ with D^0 and D^1 divisible $Z_{(p)}$-modules, and we let $e:\mathcal{U}(G) \to G$ be the kernel of the induced map from $e:\mathcal{U}(D^0) \to D^0$ to $e:\mathcal{U}(D^1) \to D^1$.

We have now constructed a right adjoint $\mathcal{U}:Z_{(p)}\text{-}Mod \to \mathcal{B}(p)$ to the forgetful functor from $\mathcal{B}(p)$ to $Z_{(p)}$-*Mod*. To show that $\mathcal{U}$ is exact we need:

LEMMA 6.3. *If $g:I \to J$ is an epimorphism of $Z_{(p)}$-modules where I is rational and J is p-torsion, then $\mathcal{U}(g):\mathcal{U}(I) \to \mathcal{U}(J)$ is also an epimorphism.*

Proof. For $y \in J$ and $s \in Z$ with $s \equiv 0 \bmod p$, we first show that the element $(y, sy, s^2y, \ldots) \in \mathcal{U}(J)$ is in the image of $\mathcal{U}(g)$. Choose $n \geq 1$ such that $p^n y = 0$, and note that r^{p-1} generates the kernel of $(Z/p^n)^* \to (Z/p)^*$. Now choose $j \in Z$ such that $r^{j(p-1)} \equiv s + 1 \bmod p^n$ and choose $x \in I$ such that $g(x) = y$. Then $\mathcal{U}(g)$ sends the element $x \in I^{j(p-1)} \subset \mathcal{U}(I)$ to $(y, sy, s^2y, \ldots) \in \mathcal{U}(J)$ as desired. Next, for $m \geq 0$ and $y \in J$, we may choose $z \in J$ with $p^{mn}z = y$ since J is the quotient of a rational group. Taking $s = p^n$ we have $s^m z = y$ and $s^{m+1}z = 0$. Thus the element $(z, \ldots, s^{m-1}z, y, 0, \ldots)$ is in the image of $\mathcal{U}(g):\mathcal{U}(I) \to \mathcal{U}(J)$, and consequently $\mathcal{U}(g)$ is onto.

PROPOSITION 6.4. *The functor $\mathcal{U}: Z_{(p)}$-Mod $\to \mathcal{B}(p)$ preserves exact sequences, arbitrary direct sums, and arbitrary direct limits.*

Proof. Since $\mathcal{U}$ is a right adjoint, it is left exact. To show that $\mathcal{U}$ is exact it suffices to show that its derived functor $R^n\mathcal{U}: Z_{(p)}$-*Mod* $\to \mathcal{B}(p)$ is zero for $n \geq 1$. For a torsion-free $Z_{(p)}$-module F, $\mathcal{U}$ preserves the exactness of $0 \to F \to F \otimes Q \to F \otimes Q/Z \to 0$ by 6.3, and thus $R^n\mathcal{U}(F) = 0$ for $n \geq 1$. For any $Z_{(p)}$-module G, there is an exact sequence $0 \to F_1 \to F_0 \to G \to 0$ where each F_i is a free $Z_{(p)}$-module. Since $R^n\mathcal{U}(F_i) = 0$ for $n \geq 1$, $R^n\mathcal{U}(G) = 0$ for $n \geq 1$. Next, $\mathcal{U}$ preserves arbitrary direct sums because it preserves direct sums of divisible groups and is exact. Finally, $\mathcal{U}$ preserves arbitrary direct limits because it preserves arbitrary direct sums and is exact.

We now construct and apply the graded version of $\mathcal{U}$.

6.5. *The functor $\mathcal{U}: \pi_* E(1)$-Mod $\to \mathcal{B}(p)_*$.* For $H \in \pi_*E(1)$-*Mod*, let $\mathcal{U}(H) \in \mathcal{B}(p)_*$ consist of the objects $\mathcal{U}(H_n) \in \mathcal{B}(p)$ for $n \in Z$ together with the unique maps $v^j: T^{j(p-1)}\mathcal{U}(H_n) \to \mathcal{U}(H_{n+2j(p-1)})$ in $\mathcal{B}(p)$ for $j \in Z$ making the diagram

$$\begin{tikzcd} T^{j(p-1)}\mathcal{U}(H_n) \arrow[r, "v^j"] \arrow[d, "e"] & \mathcal{U}(H_{n+2j(p-1)}) \arrow[d, "e"] \\ H_n \arrow[r, "v^j"] & H_{n+2j(p-1)} \end{tikzcd}$$

commute. Then $e: \mathcal{U}(H) \to H$ has the universal property that for each $X \in \mathcal{B}(p)_*$ and each $\pi_*E(1)$-homomorphism $f: X \to H$ there exists a unique map $\bar{f}: X \to \mathcal{U}(H)$ in $\mathcal{B}(p)_*$ with $e\bar{f} = f$. Thus the functor $\mathcal{U}: \pi_*E(1)$-*Mod* $\to \mathcal{B}(p)_*$ is right adjoint to the forgetful functor. It also preserves exact sequences, arbitrary direct sums, and arbitrary direct limits.

For an $E(1)$-module spectrum $Y \in \underline{Ho}^s$ the multiplication map $\mu: E(1) \wedge Y \to U$ induces a $\pi_*E(1)$-homomorphism $m: E(1)_*Y \to \pi_*Y$ and we let $\bar{m}: E(1)_*Y \to \mathcal{U}(\pi_*Y)$ be the unique map in $\mathcal{B}(p)_*$ with $e\bar{m} = m$.

PROPOSITION 6.6. *For each $E(1)$-module spectrum $Y \in \underline{Ho}^s$, the map $\bar{m}: E(1)_*Y \to \mathcal{U}(\pi_*Y)$ is an isomorphism in $\mathcal{B}(p)_*$.*

The proof depends on the following lemma. For $M \in \mathcal{B}(p)$, let M^ψ and M_ψ respectively denote the largest subobject and quotient object of M with trivial ψ-action. Thus by 5.7, M^ψ and M_ψ are respectively given by the kernel and cokernel of $\tilde{\psi}^r : M \to M$. Consequently a short exact sequence $0 \to L \to M \to N \to 0$ in $\mathcal{B}(p)$ gives rise to a six-term exact sequence

$$0 \to L^{\psi} \to M^{\psi} \to N^{\psi} \to L_{\psi} \to M_{\psi} \to N_{\psi} \to 0.$$

Note that $\mathcal{U}(G)^{\psi} \approx G$ and $\mathcal{U}(G)_{\psi} = 0$ for a torsion $Z_{(p)}$-module G.

LEMMA 6.7. *A map $f: L \to M$ in $\mathcal{B}(p)$ is an isomorphism if and only if it induces isomorphisms $L \otimes Q \approx M \otimes Q$ and $L^{\psi} \approx M^{\psi}$ and a monomorphism $L_{\psi} \to M_{\psi}$.*

Proof. The "only if" part is obvious. For the "if" part, note that ker f is torsion and $(\ker f)^{\psi} = 0$. This implies $\ker f = 0$ since $\tilde{\psi}^r$ acts locally nilpotently on each torsion object of $\mathcal{B}(p)$. Likewise $\operatorname{cok} f$ is torsion and $(\operatorname{cok} f)^{\psi} = 0$ by a six-term exact sequence argument. Thus $\operatorname{cok} f = 0$ and f is an isomorphism.

For an abelian group G and spectrum $X \in \underline{Ho}^s$, let $XG \in \underline{Ho}^s$ denote X with coefficients in G, i.e., XG is the smash product of X with a Moore spectrum of type $(G, 0)$.

6.8. *Proof of* 6.6. First, if $Y = E(1)Z/p^{\infty}$ then Y is a $K_{(p)*}$-local ($= E(1)_*$-local) torsion spectrum, and thus the sequence $Y \xrightarrow{\alpha} K_{(p)} \wedge Y \xrightarrow{\tilde{\psi}^r \wedge 1} K_{(p)} \wedge Y$ is a cofibering in $\underline{Ho}^s$ by Section 4 of [7] where, for the moment, r denotes a positive integer generating $(Z/p^2)^*$ instead of merely Γ^1. Hence the sequence $Y \xrightarrow{\alpha} E(1) \wedge Y \xrightarrow{\tilde{\psi}^r \wedge 1} E(1) \wedge Y$ is also a cofibering in $\underline{Ho}^s$ because it is a direct summand of the above sequence. Thus there is a short exact sequence $0 \to \pi_* Y \xrightarrow{\alpha} E(1)_* Y \xrightarrow{\tilde{\psi}^r} E(1)_* Y \to 0$ which is split by $m: E(1)_* Y \to \pi_* Y$. Consequently $(E(1)_* Y)^{\psi} \approx \pi_* Y$ and $(E(1)_* Y)_{\psi} = 0$, so $\bar{m}: E(1)_* Y \approx \mathcal{U}(\pi_* Y)$ by 6.7. Next, if $Y = E(1)Q$ then $E(1)_0 Y$ has a Q-basis $\{v^j w^{-j} \mid j \in Z\}$ with $\psi^k(v^j w^{-j}) = k^{j(p-1)} v^j w^{-j}$ for all $k \in Z^*_{(p)}$ and $j \in Z$, and the map $m: E(1)_0 Y \to \pi_0 Y \approx Q$ sends $v^j w^{-j}$ to 1 for all $j \in Z$. Thus, comparing $E(1)_0 Y$ with our explicit description of $\mathcal{U}(Q)$, we find $\bar{m}: E(1)_0 Y \approx \mathcal{U}(\pi_0 Y)$. Since $E(1)_n Y = 0$ when n is not divisible by $2p - 2$, we obtain $\bar{m}: E(1)_* Y \approx \mathcal{U}(\pi_* Y)$. Next, if $Y = E(1)D$ where D is a divisible $Z_{(p)}$-module, then $\bar{m}: E(1)_* Y \approx \mathcal{U}(\pi_* Y)$ since D is a direct sum of copies of Z/p^{∞} and Q. Thus, if $Y = E(1)G$ for any $Z_{(p)}$-module G, then $\bar{m}: E(1)_* Y \approx \mathcal{U}(\pi_* Y)$ since there is a cofibering $E(1)G \to E(1)D^0 \to E(1)D^1$ of $E(1)$-module spectra corresponding to an injective resolution $0 \to G \to D^1 \to D^2 \to 0$. Finally, if Y is any $E(1)$-module spectrum, then $\bar{m}: E(1)_* Y \approx \mathcal{U}(\pi_* Y)$ since Y is equivalent to the $E(1)$-module spectrum $\vee_{i=0}^{2p-3} \Sigma^i E(1)(\pi_i Y)$.

7. The Ext functors in $\mathcal{B}(p)$ and $\mathcal{B}(p)_*$. We first obtain detailed results on the functors Ext^s in the abelian category $\mathcal{B}(p)$ and then extend these results to the functors $\mathrm{Ext}^{s,t}$ in $\mathcal{B}(p)_*$ or equivalently in $\mathcal{A}(p)_*$.

PROPOSITION 7.1. *If D is a divisible $Z_{(p)}$-module, then $\mathcal{U}(D)$ is injective in $\mathcal{B}(p)$. Thus $\mathcal{B}(p)$ has enough injectives.*

Proof. $\mathcal{U}(D)$ is injective by an adjointness argument. Likewise, each $M \in \mathcal{B}(p)$ can be embedded in an injective object by choosing a $Z_{(p)}$-monomorphism $f: M \to D'$ with D' divisible and then taking the lifting $\bar{f}: M \to \mathcal{U}(D')$ in $\mathcal{B}(p)$.

For later use, we now determine all the injectives in $\mathcal{B}(p)$. For a $Z_{(p)}$-module G and $j \in Z$, let $T^{j(p-1)}G \in \mathcal{B}(p)$ equal G as a $Z_{(p)}$-module and have $\psi^k = k^{j(p-1)}$ for each $k \in Z^*_{(p)}$.

PROPOSITION 7.2. *An object $M \in \mathcal{B}(p)$ is injective if and only if*

$$M \approx \mathcal{U}(D) \oplus \Big(\bigoplus_{j \in Z} T^{j(p-1)}R_j\Big)$$

for some divisible torsion $Z_{(p)}$-module D and some rational $Z_{(p)}$-modules R_j for $j \in Z$.

Proof. The "if" part is easy, so let $M \in \mathcal{B}(p)$ be injective. As in proof 7.1, construct a monomorphism $\bar{f}: M \to \mathcal{U}(D')$ with D' divisible, and note that $\bar{f}$ has a left inverse. Thus M is divisible and its torsion subobject is injective in $\mathcal{B}(p)$ because $\mathcal{U}(D')$ has these properties. Hence $M \approx N \oplus R$ in $\mathcal{B}(p)$ where N is an injective torsion object and R is rational. Since N is a retract of $\mathcal{U}(N)$ in $\mathcal{B}(p)$ and since $\mathcal{U}(N)^\psi \approx N$ and $\mathcal{U}(N)_\psi = 0$, it follows that N^ψ is divisible and $N_\psi = 0$. Thus by 6.7 there is an isomorphism $\bar{r}: N \approx \mathcal{U}(N^\psi)$ in $\mathcal{B}(p)$ where $r: N \to N^\psi$ is a retraction. The proposition now follows using the eigenspace decomposition of $R \approx R \otimes Q$.

PROPOSITION 7.3. *For $G \in Z_{(p)}$-Mod, $L \in \mathcal{B}(p)$, and $s \geq 0$, there is a natural isomorphism* $\mathrm{Ext}^s_{\mathcal{B}(p)}(L, \mathcal{U}(G)) \approx \mathrm{Ext}^s_{Z_{(p)}}(L, G)$.

Proof. If $0 \to G \to D^0 \to D^1 \to 0$ is an injective resolution of G in $Z_{(p)}$-*Mod*, then $0 \to \mathcal{U}(G) \to \mathcal{U}(D^0) \to \mathcal{U}(D^1) \to 0$ is an injective resolution of $\mathcal{U}(G)$ in $\mathcal{B}(p)$ by 6.4 and 7.1. Thus the adjunction isomorphism $\mathrm{Hom}_{\mathcal{B}(p)}(L, \mathcal{U}(-)) \approx \mathrm{Hom}_{Z_{(p)}}(L, -)$ induces the desired natural isomorphism.

To determine $\mathrm{Ext}^s_{\mathcal{B}(p)}(L, M)$ for more general $L, M \in \mathcal{B}(p)$, we need the natural sequence

$$0 \to M \xrightarrow{\alpha} \mathcal{U}(M) \xrightarrow{\beta} \mathcal{U}(M) \xrightarrow{\gamma} M \otimes Q \to 0 \tag{7.4}$$

in $\mathcal{B}(p)$ where α is adjoint to $1: M \to M$, where $\beta = \tilde{\psi}^r - \mathcal{U}(\tilde{\psi}^r) = \psi^r - \mathcal{U}(\psi^r)$ with r a fixed integer generating Γ^1, and where γ is the composition of the canonical maps

$$\mathcal{U}(M) \to \mathcal{U}(M \otimes Q) \approx \bigoplus_{j \in Z} (M \otimes Q)^{j(p-1)} \xrightarrow{\oplus q_j} \bigoplus_{j \in Z} W_{j(p-1)} \approx M \otimes Q$$

where $M \otimes Q \approx \oplus_{j \in Z} W_{j(p-1)}$ is the eigenspace decomposition from (5.2) and where $q_j: (M \otimes Q)^{j(p-1)} \to W_{j(p-1)}$ is the projection.

LEMMA 7.5. *For each $M \in \mathcal{B}(p)$ the sequence* (7.4) *is exact.*

Proof. If M is p-torsion or rational, then the exactness of (7.4) follows using our explicit constructions of $\mathcal{U}(M)$. In general, one easily checks that $\beta\alpha = 0$ and $\gamma\beta = 0$, so (7.4) is a chain complex. Using the short exact sequences of chain complexes induced by $0 \to \tilde{M} \to M \to \bar{M} \to 0$ and $0 \to \bar{M} \to \bar{M} \otimes Q \to \bar{M} \otimes Q/Z \to 0$ where $\tilde{M}$ is the torsion subobject of M, one deduces the exactness of (7.4) from the exactness of the corresponding complexes involving $\tilde{M}$, $\bar{M} \otimes Q$, and $\bar{M} \otimes Q/Z$.

For any $Z_{(p)}$-module G, one can apply (7.4) to deduce $\mathcal{U}(G)^{\psi} \approx G$ and $\mathcal{U}(G)_{\psi} \approx G \otimes Q$ by treating G as a trivial object of $\mathcal{B}(p)$. However, our main application is:

7.6. *A spectral sequence for* $\mathrm{Ext}^s_{\mathcal{B}(p)}(L, M)$. Let $L, M \in \mathcal{B}(p)$. Using the exact sequence (7.4) and the isomorphisms 7.3, one obtains a spectral sequence $\{E_r^{j,k}\}$ with differentials $d_r: E_r^{j,k} \to E_r^{j+r, k-r+1}$ and with E_1-term given by $E_1^{0,0} = \mathrm{Hom}_{Z_{(p)}}(L, M)$, $E_1^{0,1} = \mathrm{Ext}^1_{Z_{(p)}}(L, M)$, $E_1^{1,0} = \mathrm{Hom}_{Z_{(p)}}(L, M)$, $E_1^{1,1} = \mathrm{Ext}^1_{Z_{(p)}}(L, M)$, $E_1^{2,0} = \mathrm{Hom}_{\mathcal{B}(p)}(L \otimes Q, M \otimes Q)$, and $E_1^{j,k} = 0$ otherwise. The differential $d_1: \mathrm{Ext}^k_{Z_{(p)}}(L, M) \to \mathrm{Ext}^k_{Z_{(p)}}(L, M)$ for $k = 0, 1$ is given by $d_1 f = f \circ \tilde{\psi}^r_L - \tilde{\psi}^r_M \circ f$ where $\tilde{\psi}^r_M: M \to M$ and $\tilde{\psi}^r_L: L \to L$ denote $\tilde{\psi}^r$, and the differential $d_1: \mathrm{Hom}_{Z_{(p)}}(L, M) \to \mathrm{Hom}_{\mathcal{B}(p)}(L \otimes Q, M \otimes Q)$ carries $f: L \to M$ to the map $d_1 f: L \otimes Q \to M \otimes Q$ given by the homogeneous components of $f \otimes 1: L \otimes Q \to M \otimes Q$ with respect to the eigenspace decompositions of $L \otimes Q$ and $M \otimes Q$. The only other possible differential is $d_2: E_2^{0,1} \to E_2^{2,0}$ and thus $E_3 = E_\infty$. The spectral sequence converges to $\{\mathrm{Ext}^s_{\mathcal{B}(p)}(L, M)\}$ so that there are natural exact sequences

$$0 \to E_\infty^{s,0} \to \mathrm{Ext}^s_{\mathcal{B}(p)}(L, M) \to E_\infty^{s-1,1} \to 0$$

for all s. Since $E_1^{j,k} = 0$ for $j + k > 2$, we have

PROPOSITION 7.7. *If* $s > 2$ *then* $\operatorname{Ext}^s_{\mathcal{B}(p)}(L, M) = 0$ *for all* $L, M \in \mathcal{B}(p)$.

Now suppose that $L, M \in \mathcal{B}(p)$ satisfy the condition $\operatorname{Hom}_{\mathcal{B}(p)}(L \otimes Q, M \otimes Q) = 0$ (i.e., $\psi^r \otimes 1 : L \otimes Q \to L \otimes Q$ and $\psi^r \otimes 1 : M \otimes Q \to M \otimes Q$ have no common eigenvalues). Then the spectral sequence reduces to a natural exact sequence

$$
\begin{aligned}
(7.8) \qquad 0 &\to \operatorname{Hom}_{\mathcal{B}(p)}(L, M) \xrightarrow{\rho} \operatorname{Hom}_{Z_{(p)}}(L, M) \xrightarrow{d_1} \operatorname{Hom}_{Z_{(p)}}(L, M) \\
&\xrightarrow{\sigma} \operatorname{Ext}^1_{\mathcal{B}(p)}(L, M) \xrightarrow{\rho} \operatorname{Ext}^1_{Z_{(p)}}(L, M) \xrightarrow{d_1} \operatorname{Ext}^1_{Z_{(p)}}(L, M) \\
&\xrightarrow{\sigma} \operatorname{Ext}^2_{\mathcal{B}(p)}(L, M) \to 0
\end{aligned}
$$

where ρ is the forgetful map, d_1 is as above, and $\sigma : \operatorname{Hom}_{Z_{(p)}}(L, M) \to \operatorname{Ext}^1_{\mathcal{B}(p)}(L, M)$ is as follows. For $f \in \operatorname{Hom}_{Z_{(p)}}(L, M)$, the element $\sigma(f) \in \operatorname{Ext}^1_{\mathcal{B}(p)}(L, M)$ is represented by the extension $0 \to M \xrightarrow{\epsilon} N \xrightarrow{\delta} L \to 0$ in $\mathcal{B}(p)$ such that $N = M \oplus L$ as a $Z_{(p)}$-module, $\psi^r(x, y) = (\psi^r x + fy, \psi^r y)$ for $(x, y) \in N$, $\epsilon(x) = (x, 0)$ for $x \in M$, and $\delta(x, y) = y$ for $(x, y) \in N$. To verify this description of $\sigma(f)$, consider the exact sequence $0 \to M \xrightarrow{\alpha} \mathcal{U}(M) \xrightarrow{\beta} \tilde{\mathcal{U}}(M) \to 0$ in $\mathcal{B}(p)$ where $\tilde{\mathcal{U}}(M) = \ker \gamma$. The $Z_{(p)}$-module splitting $e : \mathcal{U}(M) \to M$ induces a corresponding splitting $d : \tilde{\mathcal{U}}(M) \to \mathcal{U}(M)$, and it suffices to prove:

LEMMA 7.9. *For each* $M \in \mathcal{B}(p)$, $e\psi^r d = e : \tilde{\mathcal{U}}(M) \to M$.

Proof. First suppose M is p-torsion. Then $\tilde{\mathcal{U}}(M) = \mathcal{U}(M)$ and $d(m_1, m_2, \ldots)$ is of the form $(0, m_1, \ldots)$. Hence $e\tilde{\psi}^r d(m_1, m_2, \ldots) = m_1 = e(m_1, m_2, \ldots)$ so $e\tilde{\psi}^r d = e$, and thus $e\psi^r d = e$ because $ed = 0$. Next suppose M is finitely generated over $Z_{(p)}$. Since our conclusion holds for each $M/p^n M \in \mathcal{B}(p)$, the maps $e\psi^r d, e : \tilde{\mathcal{U}}(M) \to M$ become equal when composed with each quotient map $M \to M/p^n M$, and thus $e\psi^r d = e$ because $\bigcap_n p^n M = 0$. Now the general case follows a direct limit argument.

7.10. *The functors* $\operatorname{Ext}^{s,t}$ *in* $\mathcal{B}(p)_*$ *and* $\mathcal{A}(p)_*$. There is an equivalence of $\mathcal{B}(p)_*$ with the product of $2p - 2$ copies of $\mathcal{B}(p)$ where an object $M \in \mathcal{B}(p)_*$ goes to the objects $M_0, \ldots, M_{2p-3} \in \mathcal{B}(p)$. Thus $\mathcal{B}(p)_*$ has enough injectives and the graded extension groups in $\mathcal{B}(p)_*$ have a natural decomposition

$$\operatorname{Ext}^{s,t}_{\mathcal{B}(p)_*}(L, M) = \operatorname{Ext}^s_{\mathcal{B}(p)_*}(\Sigma^t L, M) \approx \prod_{i=0}^{2p-3} \operatorname{Ext}^s_{\mathcal{B}(p)}(L_{i-t}, M_i).$$

Moreover, the categorical equivalence $(-)^{[0]}: \mathfrak{A}(p)_* \to \mathfrak{B}(p)_*$ of 3.8 induces natural isomorphisms

$$\mathrm{Ext}^{s,t}_{\mathfrak{A}(p)_*}(B, C) \approx \mathrm{Ext}^{s,t}_{\mathfrak{B}(p)_*}(B^{[0]}, C^{[0]})$$

for $B, C \in \mathfrak{A}(p)_*$. Consequently, our results on the functors Ext^s in $\mathfrak{B}(p)$ apply to the functors $\mathrm{Ext}^{s,t}$ in $\mathfrak{B}(p)_*$ and $\mathfrak{A}(p)_*$, and these $\mathrm{Ext}^{s,t}$ vanish for $s > 2$. A slightly weaker version of this vanishing result was obtained by Adams-Baird in the equivalent category of $K_{(p)*}K_{(p)}$-comodules (see [4]).

8. The $E(1)_*$-Adams spectral sequence. For spectra $X, Y \in \underline{Ho}^s$ we construct the $E(1)_*$-Adams spectral sequence $\{E_r^{s,t}(X, Y)\}$ which has

$$E_2^{s,t}(X, Y) \approx \mathrm{Ext}^{s,t}_{\mathfrak{B}(p)_*}(E(1)_*X, E(1)_*Y)$$

and converges strongly to $[X_{E(1)}, Y_{E(1)}]_*$ where $(-)_{E(1)}$ is the $E(1)_*$-localization functor. We then observe that $E_2^{s,t}(X, Y) = 0$ for $s > 2$ and express the differential d_2 by a formula involving the $E(1)_*$-k-invariants of X and Y. This requires a discussion of $E(1)_*$-Moore spectra. We conclude by indicating the corresponding results for the $K_{(p)*}$-Adams spectral sequence. In constructing Adams spectral sequences, we follow Moss [10] since he provides composition pairings and since the method of [5] could give the wrong E_2-term when $E(1)_*X$ is not a free $\pi_*E(1)$-module.

A spectrum $Y \in \underline{Ho}^s$ is called *$E(1)_*$-injective* if $E(1)_*Y$ is injective in $\mathfrak{B}(p)_*$ and the canonical map

$$[X, Y]_* \to \mathrm{Hom}_{\mathfrak{B}(p)_*}(E(1)_*X, E(1)_*Y)$$

is an isomorphism for each $X \in \underline{Ho}^s$. Note that the $E(1)_*$-injective spectra are $E(1)_*$-local and are closed under finite wedges, retractions, and (de)suspensions in $\underline{Ho}^s$.

LEMMA 8.1. *If $Y \in \underline{Ho}^s$ is an $E(1)$-module spectrum with π_*Y divisible, or if Y is any rational spectrum, then Y is $E(1)_*$-injective.*

Proof. First let Y be an $E(1)$-module spectrum with π_*Y divisible. Using the techniques of Section 13 of [5], one shows that the canonical map $j: [X, Y] \to \mathrm{Hom}_{\pi_*E(1)}(E(1)_*X, \pi_*Y)$ is an isomorphism for each $X \in$

$\underline{Ho}^s$. Since $\pi_* Y$ is divisible and since $E(1)_* Y \approx \mathcal{U}(\pi_* Y)$ by 6.6, $E(1)_* Y$ is injective in $\mathcal{B}(p)_*$ and an adjointness argument now shows that the canonical map $[X, Y] \to \mathrm{Hom}_{\mathcal{B}(p)_*}(E(1)_* X, E(1)_* Y)$ is an isomorphism for each $X \in \underline{Ho}^s$. Thus Y is $E(1)_*$-injective. Finally, if Y is any rational spectrum, then Y is $E(1)_*$-injective because Y is a retract of the rational $E(1)$-module spectrum $E(1) \wedge Y$.

PROPOSITION 8.2. *For each injective object $M \in \mathcal{B}(p)_*$ there exists an $E(1)_*$-injective spectrum $Y \in \underline{Ho}^s$ with $E(1)_* Y \approx M$ in $\mathcal{B}(p)_*$.*

Proof. First suppose for some $n \in Z$ that $M_i = 0$ unless $i \equiv n$ mod $2p - 2$. By 7.2,

$$M_n \approx \mathcal{U}(D) \oplus \bigoplus_{j \in Z} T^{j(p-1)} R_j$$

in $\mathcal{B}(p)$ for some divisible torsion $Z_{(p)}$-module D and some rational $Z_{(p)}$-modules R_j for $j \in Z$. Thus $M \approx E(1)_* Y$ for the $E(1)_*$-injective spectrum $Y = \Sigma^n E(1) D \vee C$ where C is the rational spectrum with $\pi_i C = R_j$ when $i = n - 2j(p-1)$ and with $\pi_i C = 0$ when $i \not\equiv n$ mod $2p - 2$. The proposition now follows for any injective object $M \in \mathcal{B}(p)_*$ since M is the direct sum of $2p - 2$ objects of the above sort.

An *$E(1)_*$-Adams tower for a spectrum $Y \in \underline{Ho}^s$* is a sequence of triangles

$$Y_{n+1} \xrightarrow{i_n} Y_n \xrightarrow{j_n} J_n \to \Sigma Y_{n+1}$$

in Ho^s for $n \geq 0$ such that each J_n is $E(1)_*$-injective, each $E(1)_* i_n$ is the 0-map, and $Y_0 = Y$. Each $Y \in \underline{Ho}^s$ has an $E(1)_*$-Adams tower which may be constructed inductively by using 8.2 and the existence of enough injectives in $\mathcal{B}(p)_*$.

8.3. *The $E(1)_*$-Adams spectral sequence.* For $X, Y \in \underline{Ho}^s$ the $E(1)_*$-Adams spectral sequence $\{E_r^{s,t}(X, Y)\}_{r \geq 2}$ is obtained by taking an $E(1)_*$-Adams tower for Y, then applying $[X, -]_*$ to give an exact couple, and then taking the associated spectral sequence with the E_1-term discarded. The spectral sequence does not depend on the choice of the Adams tower and is natural in X and Y. As in [10] there is a natural isomorphism

$$E_2^{s,t}(X, Y) \approx \mathrm{Ext}^{s,t}_{\mathcal{B}(p)_*}(E(1)_* X, E(1)_* Y)$$

for each s, t, and thus $E_2^{s,t}(X, Y) = 0$ unless $s = 0, 1$, or 2. Consequently the differentials $d_r: E_r^{s,t}(X, Y) \to E_r^{s+r,t+r-1}(X, Y)$ are all trivial except possibly for $d_2: E_2^{0,t}(X, Y) \to E_2^{2,t+1}(X, Y)$. Thus $E_\infty^{s,t}(X, Y) = E_3^{s,t}(X, Y)$ where $E_\infty^{s,t}(X, Y)$ denotes $\bigcap_r E_r^{s,t}(X, Y)$. For $s \geq 0$ let $F^s[X, Y]_*$ denote the image of the map $[X, Y_s]_* \to [X, Y]_*$ given by an $E(1)_*$-Adams tower for Y, and note that $\{F^s[X, Y]_*\}_{s \geq 0}$ is a natural decreasing filtration of $[X, Y]_* = F^0[X, Y]_*$. In fact, an element $f \in [X, Y]_*$ lies in $F^s[X, Y]_*$ if and only if f is expressible as an s-fold composite $f = f_1 \cdots f_s$ where each $E(1)_* f_i$ is 0. In general there is a natural injection

$$h: (F^s/F^{s+1})[X, Y]_{t-s} \to E_\infty^{s,t}(X, Y)$$

which is induced for $s = 0$ by the canonical map $[X, Y]_* \to \mathrm{Hom}_{\mathcal{B}(p)_*}(E(1)_* X, E(1)_* Y)$.

PROPOSITION 8.4. *For $X, Y \in \underline{Ho}^s$ with Y $E(1)_*$-local, the spectral sequence $\{E_r^{s,t}(X, Y)\}$ converges strongly to $[X, Y]_*$ in the sense that $F^s[X, Y]_* = 0$ for $s > 2$ and*

$$h: (F^s/F^{s+1})[X, Y]_{t-s} \approx E_\infty^{s,t}(X, Y) = E_3^{s,t}(X, Y)$$

for each s, t.

Proof. Let $\{Y_{n+1} \to Y_n \to J_n \to \Sigma Y_{n+1}\}_{n \geq 0}$ be an $E(1)_*$-Adams tower for Y. Then each J_n is $E(1)_*$-local since it is $E(1)_*$-injective, and each Y_n is $E(1)_*$-local by induction on n (see 1.4 of [7]). Furthermore, $E(1)_* Y_2$ is injective in $\mathcal{B}(p)_*$ since $E(1)_* Y$ has injective dimension ≤ 2. Thus, by 8.2 there exists an $E(1)_*$-equivalence $g: Y_2 \to L$ in $\underline{Ho}^s$ such that L is $E(1)_*$-injective. Now $g: Y_2 \simeq L$ since g is an $E(1)_*$-equivalence of $E(1)_*$-local spectra. Thus Y_2 is $E(1)_*$-injective, and we may suppose that our $E(1)_*$-Adams tower for Y has $Y_2 = J_2$ and $Y_n = 0 = J_n$ for all $n \geq 3$. With this tower, the proposition follows immediately.

In general, the "actual" target of the spectral sequence is $[X_{E(1)}, Y_{E(1)}]_* \approx [X, Y_{E(1)}]_*$ because of the following easy result:

PROPOSITION 8.5. *For $X, Y \in \underline{Ho}^s$ the spectral sequence $\{E_r^{s,t}(X, Y)\}$ is naturally isomorphic to $\{E_r^{s,t}(X_{E(1)}, Y_{E(1)})\}$ and thus converges strongly to $[X_{E(1)}, Y_{E(1)}]_*$ in the sense of* 8.4.

8.6. *Composition pairings.* Theorem 2.1 of [10] applies (in dualized form) to the $E(1)_*$-Adams spectral sequence and shows that, for $W, X, Y \in \underline{Ho}^s$ and $2 \leq r \leq \infty$, there are natural associative pairings

$$E_r^{s,t}(X, Y) \otimes E_r^{u,v}(W, X) \to E_r^{s+u,t+v}(W, Y)$$

with the following properties. For $r = 2$, the pairing is given by the Yoneda pairing

$$\operatorname{Ext}^{s,t}_{\mathcal{B}(p)_*}(E(1)_*X, E(1)_*Y) \otimes \operatorname{Ext}^{u,v}_{\mathcal{B}(p)_*}(E(1)_*W, E(1)_*X)$$

$$\to \operatorname{Ext}_{\mathcal{B}(p)_*}^{s+u,t+v}(E(1)_*W, E(1)_*Y).$$

If $a \in E_2^{s,t}(X, Y)$ and $b \in E_r^{u,v}(W, X)$, then $d_r(ab) = (d_r a)b + (-1)^{t-s}a(d_r b)$. The pairings commute with the isomorphisms $E_{r+1} \approx H(E_r)$ and $E_\infty \approx \bigcap_r E_r$. The composition pairing $[X, Y]_* \otimes [W, X]_* \to [W, Y]_*$ preserves filtration (i.e., if $a \in F^s[X, Y]_*$ and $b \in F^u[W, X]_*$ then $ab \in F^{s+u}[W, Y]_*$), and the induced pairing of filtration quotients agrees with the given pairing of E_∞-terms.

Before giving our formula for the differential d_2 we must introduce the $E(1)_*$-Moore spectra and define the $E(1)_*$-k-invariant. By an $E(1)_*$-*Moore spectrum* of type (M, n) for $M \in \mathcal{B}(p)$ and $n \in Z$ we mean an $E(1)_*$-local spectrum $Y \in \underline{Ho}^s$ together with an isomorphism $E(1)_n Y \approx M$ in $\mathcal{B}(p)$ such that $E(1)_i Y = 0$ for each $i \neq n \bmod 2p - 2$.

PROPOSITION 8.7. *For each $M \in \mathcal{B}(p)$ and $n \in Z$ there exists an $E(1)_*$-Moore spectrum of type (M, n). If X and Y are $E(1)_*$-Moore spectra of types (M, n) and (N, n) respectively, then $E(1)_n : [X, Y] \approx \operatorname{Hom}_{\mathcal{B}(p)}(M, N)$.*

Proof. Choose an injective resolution $0 \to M \to I_0 \to I_1 \to I_2 \to 0$ for M in $\mathcal{B}(p)$, and then apply 8.2 to give $E(1)_*$-injective spectra $J_0, J_1, J_2 \in \underline{Ho}^s$ such that each J_i is in $E(1)_*$-Moore spectrum of type (I_i, n). Let $F_1 \in \underline{Ho}^s$ be the homotopy fibre of a map $J_0 \to J_1$ which is carried by $E(1)_n$ to $I_0 \to I_1$. Next let $F_2 \in \underline{Ho}^s$ be the homotopy fibre of a map $F_1 \to \Sigma^{-1} J_2$ which is carried by $E(1)_{n-1}$ to an isomorphism. Then F_2 is an $E(1)_*$-Moore spectrum of type (M, n). The last statement of 8.7 may be proved using the $E(1)_*$-Adams spectral sequence.

For each $M \in \mathcal{B}(p)$ and $n \in Z$, let $\mathfrak{M}(M, n) \in \underline{Ho}^s$ denote an $E(1)_*$-Moore spectrum of type (M, n), and note that it is natural in M and unique up to a canonical equivalence. Also note that $\mathfrak{M}(M, n) \simeq \mathfrak{M}(T^{j(p-1)}M, n + 2j(p-1))$ for each $j \in Z$. More generally, for each $A \in \mathcal{B}(p)_*$ let $\mathfrak{M}(A) \in \underline{Ho}^s$ denote the $E(1)_*$-local spectrum $\vee_{n=0}^{2p-3} \mathfrak{M}(A_n, n)$, and for each map $\varphi: A \to B$ in $\mathcal{B}(p)_*$ let $\mathfrak{M}(\varphi): \mathfrak{M}(A) \to \mathfrak{M}(B)$ be the induced map $\vee_{n=0}^{2p-3} \mathfrak{M}(\varphi_n)$ in $\underline{Ho}^s_{E(1)}$. Now $\mathfrak{M}: \mathcal{B}(p)_* \to \underline{Ho}^s$ is a functor and there is a natural isomorphism $E(1)_*\mathfrak{M}(A) \approx A$ for $A \in \mathcal{B}(p)_*$. By a *generalized* $E(1)_*$-*Moore spectrum* we mean a spectrum equivalent to $\mathfrak{M}(A)$ for some $A \in \mathcal{B}(p)_*$.

LEMMA 8.8. *If* $X, Y \in \underline{Ho}^s$ *are generalized* $E(1)_*$-*Moore spectra, then* $d_2 = 0$ *in the* $E(1)_*$-*Adams spectral sequence* $\{E_r(X, Y)\}$.

Proof. We may suppose $X = \mathfrak{M}(A)$ and $Y = \mathfrak{M}(B)$ for $A, B \in \mathcal{B}(p)_*$. Then $d_2 = 0$ since the map $E(1)_*: [X, Y]_* \to \mathrm{Hom}_{\mathcal{B}(p)_*}(E(1)_*X, E(1)_*Y)_*$ is onto because it has a right inverse induced by the functor $\mathfrak{M}$.

8.9. *The* $E(1)_*$-k-*invariant*. For $Y \in \underline{Ho}^s$ we construct the $E(1)_*$-k-*invariant*

$$k_Y \in E_2^{2,1}(Y, Y) \approx \mathrm{Ext}^{2,1}_{\mathcal{B}(p)_*}(E(1)_*Y, E(1)_*Y)$$

by first choosing a generalized $E(1)_*$-Moore spectrum $Y' \in \underline{Ho}^s$ with isomorphism $\alpha: E(1)_*Y' \approx E(1)_*Y$ and then letting $k_Y = (d_2\alpha)\alpha^{-1}$ using $\alpha \in E_2^{0,0}(Y', Y)$ and $\alpha^{-1} \in E_2^{0,0}(Y, Y')$. To show that k_Y is well-defined, let $Y'' \in \underline{Ho}^s$ be another generalized $E(1)_*$-Moore spectrum with $\beta: E(1)_*Y'' \approx E(1)_*Y$, and take $\gamma: E(1)_*Y'' \approx E(1)_*Y'$ such that $\beta = \alpha\gamma$. Since $d_2\gamma = 0$ we have $(d_2\beta)\beta^{-1} = (d_2\alpha)\gamma\gamma^{-1}\alpha^{-1} = (d_2\alpha)\alpha^{-1}$ as desired. The element k_Y has components

$$k_Y(n, n+1) \in \mathrm{Ext}^2_{\mathcal{B}(p)}(E(1)_nY, E(1)_{n+1}Y)$$

for $n \in Z$, and is determined by $2p - 2$ successive components. If Y is a generalized $E(1)_*$-Moore spectrum, then $k_Y = 0$ by 8.8.

PROPOSITION 8.10. *For* $X, Y \in \underline{Ho}^s$ *the differential* $d_2: E_2^{0,t}(X, Y) \to E_2^{2,t+1}(X, Y)$ *in the* $E(1)_*$-*Adams spectral sequence is given by* $d_2f = k_Yf + (-1)^{t+1}fk_X$ *for each* $f \in E_2^{0,t}(X, Y)$.

Proof. Let X' and Y' be generalized $E(1)_*$-Moore spectra with isomorphisms $\alpha: E(1)_*X' \approx E(1)_*X$ and $\beta: E(1)_*Y' \approx E(1)_*Y$. Let $f' \in E_2^{0,t}(X', Y')$ be such that $\beta f'\alpha^{-1} = f \in E_2^{0,t}(X, Y)$, where we take $\alpha \in E_2^{0,0}(X', X)$, $\alpha^{-1} \in E_2^{0,0}(X, X')$, and $\beta \in E_2^{0,0}(Y', Y)$. Since $d_2f' = 0$, we have

$$\begin{aligned} d_2f = d_2(\beta f'\alpha^{-1}) &= (d_2\beta)f'\alpha^{-1} + (-1)^t\beta f'(d_2\alpha^{-1}) \\ &= (d_2\beta)\beta^{-1}\beta f'\alpha^{-1} + (-1)^t\beta f'\alpha^{-1}\alpha(d_2\alpha^{-1}) \\ &= k_Yf + (-1)^t f\alpha(d_2\alpha^{-1}) \end{aligned}$$

and the proposition now follows because

$$\alpha(d_2\alpha^{-1}) = d_2(\alpha\alpha^{-1}) - (d_2\alpha)\alpha^{-1} = d_2(1) - k_X = -k_X.$$

COROLLARY 8.11. *For X, $Y \in \underline{Ho}^s$ with Y $E(1)_*$-local, a homomorphism $f \in \mathrm{Hom}_{\mathcal{B}(p)_*}(E(1)_*X, E(1)_*Y)_t$ is induced by some map in $[X, Y]_t$ if and only if $k_Y f = (-1)^t f k_X$.*

Proof. This follows by combining 8.10 and 8.4.

8.12. *The corresponding results over $K_{(p)*}$.* Recall from 3.8 and 4.2 that there is a categorical equivalence $(-)^{[0]}: \mathcal{A}(p)_* \to \mathcal{B}(p)_*$ and a natural isomorphism $E(1)_*X \approx K_*(X; Z_{(p)})^{[0]}$ for $X \in Ho^s$. Thus the $E(1)_*$-Adams spectral sequence $\{E_r^{s,t}(X, Y)\}$ with $E_2^{s,t}(X, Y) \approx \mathrm{Ext}^{s,t}_{\mathcal{B}(p)*}(E(1)_*X, E(1)_*Y)$ for X, $Y \in \underline{Ho}^s$ corresponds isomorphically to the $K_{(p)*}$-Adams spectral sequence $\{E_r^{s,t}(X, Y; K_{(p)*})\}$ with $E_2^{s,t}(X, Y; K_{(p)*}) \approx \mathrm{Ext}^{s,t}_{\mathcal{A}(p)_*}(K_{(p)*}X, K_{(p)*}X)$. Indeed, the two spectral sequences can be constructed identically since an $E(1)_*$-Adams tower for Y is the same as a $K_{(p)*}$-Adams tower for Y. The correspondence of E_2-terms is given algebraically by the natural isomorphisms

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(p)_*}(K_{(p)*}X, K_{(p)*}Y) \approx \mathrm{Ext}^{s,t}_{\mathcal{B}(p)_*}(E(1)_*X, E(1)_*Y)$$

of 7.11, 4.2. Moreover, these correspondences respect compositions and Yoneda products. Finally, for $W \in \underline{Ho}^s$, there is a $K_{(p)*}$-k-invariant

$$k_W \in E_2^{2,1}(W, W; K_{(p)*}) \approx \mathrm{Ext}^{2,1}_{\mathcal{A}(p)_*}(K_{(p)*}W, K_{(p)*}W)$$

corresponding to the $E(1)_*$-k-invariant and such that $d_2 f = k_Y f + (-1)^{t+1} f k_X$ for $f \in E_2^{0,t}(X, Y; K_{(p)*})$. Thus when Y is $E(1)_*$-local ($= K_{(p)*}$-local), a homomorphism $f \in \mathrm{Hom}_{\mathcal{A}(p)_*}(K_{(p)*}X, K_{(p)*}Y)_t$ is induced by some map in $[X, Y]_t$ if and only if $k_Y f = (-1)^t f k_X$.

9. An algebraic classification of $E(1)_*$-local spectra. Let $\underline{Ho}^s_{E(1)}$ denote the homotopy category of $E(1)_*$-local ($= K_{(p)*}$-local) spectra, and recall that $\underline{Ho}^s_{E(1)}$ is equivalent to the category of fractions obtained from $\underline{Ho}^s$ by giving formal inverses to the $E(1)_*$-equivalences ($= K_{(p)*}$-equivalences). Using $E(1)_*$-k-invariants, we now algebraically determine all the homotopy types in $\underline{Ho}^s_{E(1)}$ and determine the bigraded category obtained from $\underline{Ho}^s_{E(1)}$ by taking Adams filtration quotients. We also determine which homotopy types in $\underline{Ho}^s_{E(1)}$ may be obtained as $E(1)_*$-localizations of finite CW-spectra. We conclude by indicating the corresponding results over $K_{(p)*}$.

Let $k\mathcal{B}(p)_*$ denote the additive category such that an object of $k\mathcal{B}(p)_*$ is a pair (M, κ) with $M \in \mathcal{B}(p)_*$ and $\kappa \in \mathrm{Ext}^{2,1}_{\mathcal{B}(p)_*}(M, M)$ and a morphism from (M, κ) to (N, λ) in $k\mathcal{B}(p)_*$ is a map $f: M \to N$ in $\mathcal{B}(p)_*$ with $\lambda f = f\kappa$ in $\mathrm{Ext}^{2,1}_{\mathcal{B}(p)_*}(M, N)$. By 8.11 there is a full additive functor $kE(1): \underline{Ho}^s_{E(1)} \to k\mathcal{B}(p)_*$ sending each $Y \in \underline{Ho}^s_{E(1)}$ to $(E(1)_*Y, k_Y)$ where k_Y is the $E(1)_*$-k-invariant of Y.

THEOREM 9.1. *For each $(M, \kappa) \in k\mathcal{B}(p)_*$ there exists $Y \in \underline{Ho}^s_{E(1)}$ with $(E(1)_*Y, k_Y) \approx (M, \kappa)$ in $k\mathcal{B}(p)_*$. Thus the homotopy types in $\underline{Ho}^s_{E(1)}$ correspond to the isomorphism classes in $k\mathcal{B}(p)_*$.*

Proof. The last statement follows from the first since any isomorphism $kE(1)_*X \approx kE(1)_*Y$ in $k\mathcal{B}(p)_*$ must be induced by an $E(1)_*$-equivalence $X \to Y$ in $\underline{Ho}^s_{E(1)}$ which is automatically a homotopy equivalence. To prove the first, let $\mathfrak{M}: \mathcal{B}(p)_* \to Ho^s$ be as in Section 8. Given $(M, \kappa) \in k\mathcal{B}(p)_*$ form a short exact sequence $0 \to M \xrightarrow{\alpha} I \xrightarrow{\beta} N \to 0$ in $\mathcal{B}(p)_*$ with I injective, and let $f: \mathfrak{M}(I) \to \mathfrak{M}(N)$ be a map (to be specified later) in $\underline{Ho}^s_{E(1)}$ such that $f_* = 0: E(1)_*\mathfrak{M}(I) \to E(1)_*\mathfrak{M}(N)$. Let $Y_f \in \underline{Ho}^s_{E(1)}$ be the homotopy theoretic fibre of $f + \mathfrak{M}(\beta): \mathfrak{M}(I) \to \mathfrak{M}(N)$ and note that it fits in a triangle

$$\Sigma^{-1}\mathfrak{M}(N) \to Y_f \to \mathfrak{M}(I) \xrightarrow{f+\mathfrak{M}(\beta)} \mathfrak{M}(N). \tag{9.2}$$

Identifying $E(1)_*Y_f$ with M in the obvious way, it will suffice to select f so that $k_{Y_f} = \kappa$ in $\mathrm{Ext}^{2,1}_{\mathcal{B}(p)_*}(M, M)$. Form a short exact sequence $0 \to N \xrightarrow{\gamma} I' \xrightarrow{\delta} I'' \to 0$ in $\mathcal{B}(p)_*$ with I' and I'' injective. Now construct an $E(1)_*$-Adams tower for Y_f such that the triangle $Y_{n+1} \to Y_n \to J_n \to \Sigma Y_{n+1}$ is given by (9.2) for $n = 0$, by

$$\Sigma^{-2}\mathfrak{M}(I'') \to \Sigma^{-1}\mathfrak{M}(N) \xrightarrow{\Sigma^{-1}\mathfrak{M}(\gamma)} \Sigma^{-1}\mathfrak{M}(I') \xrightarrow{\Sigma^{-1}\mathfrak{M}(\delta)} \Sigma^{-1}\mathfrak{M}(I'')$$

for $n = 1$, by $0 \to \Sigma^{-2}\mathfrak{M}(I'') \xrightarrow{1} \Sigma^{-2}\mathfrak{M}(I'') \to 0$ for $n = 2$, and by $0 \to 0 \to 0 \to 0$ for $n \geq 3$. This tower realizes the injective resolution $0 \to M \xrightarrow{\alpha} I \xrightarrow{\gamma\beta} I' \xrightarrow{\delta} I'' \to 0$ of M in $\mathcal{B}(p)_*$. Let Y' denote $\mathfrak{M}(M) \in \underline{Ho}^s$. The $E(1)_*$-k-invariant of Y_f in $\mathrm{Ext}^{2,1}_{\mathcal{B}(p)_*}(M, M)$ corresponds to an image element of $\mathfrak{M}(\alpha) \in [Y', J_0]_0$ in $\mathrm{Hom}_{\mathcal{B}(p)_*}(M, I'')_1$ under the composite relation

$$[Y', J_0]_0 \xrightarrow{\partial} [Y', Y_1]_{-1} \leftarrow [Y', Y_2]_{-1} \xrightarrow{\approx} [Y', J_2]_{-1} \approx \mathrm{Hom}_{\mathcal{B}(p)*}(M, I'')_1 .$$

Going in the reverse direction, choose $\theta \in \mathrm{Hom}_{\mathcal{B}(p)_*}(M, I'')_1$ corresponding to the given $\kappa \in \mathrm{Ext}^{2,1}_{\mathcal{B}(p)*}(M, M)$ and let $\bar{\theta} \in [Y', Y_1]_{-1}$ denote the image element of θ. It will suffice to select $f \in F^1[J_0, Y_1]_{-1}$ so that the composite of $M(\alpha) \in [Y', J_0]_0$ with $f + \mathfrak{M}(\beta) \in [J_0, Y_1]_{-1}$ equals $\bar{\theta} \in [Y', Y_1]_{-1}$. Note that $\mathfrak{M}(\beta)\mathfrak{M}(\alpha) = \mathfrak{M}(\beta\alpha) = 0$ and that $\bar{\theta}$ is in $F^1[Y', Y_1]_{-1}$ because $E(1)_*i_1 : E(1)_*Y_2 \to E(1)_*Y_1$ is zero. Thus it will suffice to select an f which is carried to $\bar{\theta}$ by $M(\alpha)^* : F^1[J_0, Y_1]_{-1} \to F^1[Y', Y_1]_{-1}$. This can be done since $E(1)_*Y_1$ has injective dimension ≤ 1, and thus $M(\alpha)^*$ corresponds to the epimorphism

$$\mathrm{Ext}^{1,0}_{\mathcal{B}(p)_*}(E(1)_*J_0, E(1)_*Y_1) \to \mathrm{Ext}^{1,0}_{\mathcal{B}(p)_*}(E(1)_*Y', E(1)_*Y_1)$$

induced by the monomorphism $M(\alpha)_* : E(1)_*Y' \to E(1)_*J_0$.

9.3. *The bigraded categories $\underline{GrHo}^s_{E(1)}$ and $k\mathcal{B}(p)_{**}$.* For $X, Y \in \underline{Ho}^s_{E(1)}$ let $\{F_s[X, Y]_*\}_{s\geq 0}$ be the $E(1)_*$-Adams filtration and recall that $F_s[X, Y]_* = 0$ for $s \geq 3$. Let $\underline{GrHo}^s_{E(1)}$ be the associated bigraded additive category whose objects are those of $\underline{Ho}^s_{E(1)}$ and whose morphism groups $[X, Y]_{s,t}$ are given by $(F_s/F_{s+1})[X, Y]_{t-s}$. For $(L, \lambda), (M, \mu) \in k\mathcal{B}(p)_*$ and $q \in Z$, define $d_2 : \mathrm{Hom}_{\mathcal{B}(p)*}(L, M)_q \to \mathrm{Ext}^{2,q+1}_{\mathcal{B}(p)*}(L, M)$ by $d_2 f = \mu f + (-1)^{q+1} f\lambda$. Let $k\mathcal{B}(p)_{**}$ be the bigraded additive category whose objects are those of $k\mathcal{B}(p)_*$ and whose morphism groups $[(L, \lambda), (M, \mu)]_{s,t}$ are given by the kernel of $d_2 : \mathrm{Hom}_{\mathcal{B}(p)*}(L, M)_t \to \mathrm{Ext}^{2,t+1}_{\mathcal{B}(p)*}(L, M)$ for $s = 0$, by $\mathrm{Ext}^{1,t}_{\mathcal{B}(p)*}(L, M)$ for $s = 1$, by the cokernel of $d_2 : \mathrm{Hom}_{\mathcal{B}(p)*}(L, M)_{t-1} \to \mathrm{Ext}^{2,t}_{\mathcal{B}(p)*}(L, M)$ for $s = 2$, and by zero for $s > 2$. The composition in $k\mathcal{B}(p)_{**}$ is induced by the Yoneda product in the obvious way. Theorem 9.1 and the results of Section 8 easily imply

THEOREM 9.4. *There is an additive equivalence $kE(1)_{**} : \underline{GrHo}^s_{E(1)} \to k\mathcal{B}(p)_{**}$ sending each $X \in \underline{GrHo}^s_{E(1)}$ to $(E(1)_*X, k_X) \in k\mathcal{B}(p)_{**}$.*

Remark. In general, for $X, Y \in \underline{Ho}^s_{E(1)}$ the group $[X, Y]_n$ need not split into $\oplus_{s=0}^2 [X, Y]_{s,n+s}$. For instance, let Y be a spectrum such that $E(1)_nY \approx Z/p$ for $n \equiv 0, 1 \bmod 2p - 2$, $E(1)_nY = 0$ otherwise, and $k_Y \neq 0$. We can show that $[S_{E(1)}, Y]_0 \approx \pi_0 Y \approx Z/p^2$ while $[S_{E(1)}, Y]_{s,s} \approx Z/p$ for $s = 0, 1$. Of course, this difficulty disappears in

9.5. *The homotopy category of generalized $E(1)_*$-Moore spectra.* Since a generalized $E(1)_*$-Moore spectrum $\mathfrak{M}(M) \in \underline{Ho}^s_{E(1)}$ is constructed as a wedge $\vee_{n=0}^{2p-3} \mathfrak{M}(M_n, n)$, one easily obtains canonical isomorphisms

$$[\mathfrak{M}(M), \mathfrak{M}(N)]_n \approx \bigoplus_{s=0}^{2} \operatorname{Ext}^{s,s+n}_{\mathcal{B}(p)_*}(M, N)$$

for $M, N \in \mathcal{B}(p)_*$ such that compositions of homotopy classes correspond to Yoneda products. Thus the homotopy category of generalized $E(1)_*$-Moore spectra is equivalent to the algebraic category having the same objects as $\mathcal{B}(p)_*$ and having morphism groups given by the above sums of Ext's with compositions given by Yoneda products. We remark that an $E(1)_*$-local spectrum Y is automatically a generalized $E(1)_*$-Moore spectrum if the groups $\operatorname{Ext}^2_{\mathcal{B}(p)}(E(1)_n Y, E(1)_{n+1} Y)$ vanish for all n because this implies $k_Y = 0$. This happens, for instance, when the groups $E(1)_* Y$ vanish in all even (or odd) dimensions, or when they all have injective dimensions ≤ 1 in $\mathcal{B}(p)$. Furthermore, for an $E(1)_*$-local spectrum Y whose groups $E(1)_* Y$ are all free Z/p^n-modules for some fixed n, we can shows that Y is a generalized $E(1)_*$-Moore spectrum if and only if $p^n = 0 : Y \to Y$. Thus, for instance, XZ/p is a generalized $E(1)_*$-Moore spectrum for each $E(1)_*$-local spectrum X. We hope to discuss these and other splittings in a future note.

We next analyze the full subcategory $\underline{Ho}^s_{fE(1)}$ of $\underline{Ho}^s_{E(1)}$ given by all $E(1)_*$-localizations of finite CW-spectra. The following lemma easily implies that *$\underline{Ho}^s_{fE(1)}$ is equivalent to the category of fractions obtained from $\underline{Ho}^s_f$ by giving formal inverses to its $E(1)_*$-equivalences, where $\underline{Ho}^s_f$ is the homotopy category of finite CW-spectra.*

LEMMA 9.6. *For each $B \in \underline{Ho}^s_f$ there exists a sequence $B = B_0 \to B_1 \to B_2 \to \cdots$ of $E(1)_*$-equivalences in $\underline{Ho}^s_f$ whose homotopy direct limit is the $E(1)_*$-localization $B_{E(1)}$. Thus for each $W \in \underline{Ho}^s_f$,* $\operatorname{colim}_n [W, B_n]_* \approx [W, B_{E(1)}]_*$.

Proof. By 2.11 and 4.8 of [7], a spectrum X is $E(1)_*$-local if and only if $[MZ/q, X]_* = 0$ for each prime $q \neq p$ and $[V(1), X]_* = 0$, where MZ/q is the Moore spectrum $S^0 \cup_q e^1$ and where $V(1)$ is the cofibre of Adams' K_*-equivalence $\Sigma^{2p-2} MZ/p \to MZ/p$. Let $\{L_n\}_{n \geq 0}$ be an indexing of the countable collection of spectra given by all $\Sigma^i\, MZ/q$ and $\Sigma^i\, V(1)$ for $i \in Z$ and primes $q \neq p$. Let $B_0 = B$ and suppose inductively that the finite CW-spectrum B_n is given. Let F_n denote the $E(1)_*$-acyclic finite CW-spectrum $\vee_{i=0}^n \vee_f L_{i,f}$ where f ranges over the finite set $[L_i, B_n]$ and where $L_{i,f}$ is a copy of L_i. Then construct $B_n \to B_{n+1}$ as the homotopy cofibre of a map $F_n \to B_n$ acting by f on each $L_{i,f}$. The homotopy direct limit of the resulting sequence $B = B_0 \to B_1 \to B_2 \to \cdots$ is $E(1)_*$-local by the above criterion, and the lemma follows easily.

THEOREM 9.7. *If $Y \in \underline{Ho}^s_{E(1)}$ is an $E(1)_*$-local spectrum with each $E(1)_n Y$ finitely generated over $Z_{(p)}$, then $Y \simeq W_{E(1)}$ for some finite CW-spectrum W.*

This implies that an object (M, κ) of $k\mathcal{B}(p)_*$ corresponds to an object of $\underline{Ho}^s_{fE(1)}$ if and only if each M_n is finitely generated over $Z_{(p)}$. We prove it in 9.10 using two lemmas:

LEMMA 9.8. *If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is a triangle in $\underline{Ho}^s_{E(1)}$ and if two of the three spectra X, Y, Z are in $\underline{Ho}^s_{fE(1)}$, then so is the third.*

Proof. We may suppose that there are $E(1)_*$-localizations $\eta : A \to X$ and $\eta : B \to Y$ with $A, B \in \underline{Ho}^s_f$. Then by 9.6 there exists a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B_n \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \nu} \\ X & \xrightarrow{u} & Y \end{array}$$

in $\underline{Ho}^s$ such that $B_n \in \underline{Ho}^s_f$ and ν is an $E(1)_*$-localization. Thus Z is in $\underline{Ho}^s_{fE(1)}$ because it is the $E(1)_*$-localization of the homotopy theoretic cofibre of α.

LEMMA 9.9. *For any $M, N \in \mathcal{B}(p)$ with M finite and any $e \in \mathrm{Ext}^2_{\mathcal{B}(p)}(M, N)$, there exists an epimorphism $\theta : L \to M$ in $\mathcal{B}(p)$ with L finite and such that $\theta^*(e) = 0$ in $\mathrm{Ext}^2_{\mathcal{B}(p)}(L, N)$.*

Proof. Letting $I^0 = (N \otimes Q) \oplus \mathcal{U}(D)$ where D is a divisible torsion $Z_{(p)}$-module containing the torsion submodule of N, construct an injective resolution $0 \to N \to I^0 \to I^1 \to I^2 \to 0$ of N in $\mathcal{B}(p)$ with I^1 and I^2 both torsion. Then choose $\epsilon : M \to I^2$ representing e and form the pull-back square

$$\begin{array}{ccc} P & \xrightarrow{\theta} & M \\ \downarrow & & \downarrow{\scriptstyle \epsilon} \\ I^1 & \longrightarrow & I^2 \end{array}$$

in $\mathcal{B}(p)$. Note that P is a torsion object and $\theta: P \to M$ is onto. Thus, since M is finite, there is a finite subobject $L \subset P$ with $\theta(L) = M$. Now $\theta: L \to M$ has the desired properties.

9.10. *Proof of* 9.7. The total rational dimension of $Q \otimes \pi_* Y$ is finite. Thus there is a triangle $V \to Y \to W \to \Sigma V$ in $\underline{Ho}^s_{E(1)}$ such that V is a finite wedge of $E(1)_*$-localized sphere spectra and W is a torsion spectrum with each $E(1)_n W$ finite. By 9.8, it suffices to show that W is in $\underline{Ho}^s_{fE(1)}$. Using 9.9, choose an epimorphism $\theta: L \to E(1)_{2p-2} W$ in $\mathcal{B}(p)$ with L finite and such that $\theta^* k_W(2p-2, 2p-1) = 0$ in $\mathrm{Ext}^2_{\mathcal{B}(p)}(L, E(1)_{2p-1} W)$. Then choose a map $\mathfrak{M}(L, 2p-2) \to W$ carried to θ by $E(1)_{2p-2}$, and let $W_{2p-1} \in \underline{Ho}^s$ denote its homotopy cofibre. Proceeding downward, given $W_n \in \underline{Ho}^s$ with $E(1)_i W_n = 0$ for $n < i \leq 2p-2$, choose a map $\mathfrak{M}(E(1)_n W_n, n) \to W_n$ carried to the identity by $E(1)_n$, and let W_{n-1} denote its homotopy cofibre. We obtain an "$E(1)_*$-Postnikov tower" of spectra $W_{2p-1}, \ldots, W_1, W_0$ with $W_0 = 0$. By 9.8 it now suffices to show that $\mathfrak{M}(N, n)$ is in $\underline{Ho}^s_{fE(1)}$ for each finite $N \in \mathcal{B}(p)$ and $n \in Z$. This follows when N has trivial ψ-action because $\mathfrak{M}(N, n)$ is then the $E(1)_*$-localization of the corresponding ordinary Moore spectrum. It follows in general, using 9.8 and Section 5, by noting that N has a finite filtration (e.g., given by kernels of $(\tilde{\psi}^r)^i: N \to N$ for $i \geq 0$) such that the filtration quotients have trivial ψ-action.

9.11. *The corresponding results over $K_{(p)*}$.* Let $k\mathcal{A}(p)_*$ denote the additive category such that an object of $k\mathcal{A}(p)_*$ is a pair (M, κ) with $M \in \mathcal{A}(p)_*$ and $\kappa \in \mathrm{Ext}^{2,1}_{\mathcal{A}(p)_*}(M, M)$, and a morphism from (M, κ) to (N, λ) in $k\mathcal{A}(p)_*$ is a map $f: M \to N$ in $\mathcal{A}(p)_*$ with $\lambda f = f\kappa$ in $\mathrm{Ext}^{2,1}_{\mathcal{A}(p)_*}(M, N)$. By 8.12 there is a full additive functor $kK_{(p)*}: \underline{Ho}^s_{E(1)} \to k\mathcal{A}(p)_*$ sending each $Y \in \underline{Ho}^s_{E(1)}$ to $(K_{(p)*} Y, k_Y)$ where k_Y is the $K_{(p)*}$-k-invariant of Y. Theorem 9.1 implies that $kK_{(p)*}$ induces a correspondence between the homotopy types in $\underline{Ho}^s_{E(1)}$ and the isomorphism classes in $k\mathcal{A}(p)_*$. Theorem 9.4 implies that $kK_{(p)*}$ induces an equivalence between the bigraded categories $\underline{GrHo}^s_{E(1)}$ and $k\mathcal{A}(p)_{**}$. Finally, Theorem 9.7 implies than an object $(M, \kappa) \in k\mathcal{A}(p)_*$ corresponds to the $E(1)_*$-localization ($= K_{(p)*}$-localization) of a finite CW-spectrum if and only if each M_n is finitely generated over $Z_{(p)}$.

10. An interpretation of $\mathcal{B}(p)_*$ and $\mathcal{A}(p)_*$ as categories of comodules. Having previously shown that $\mathcal{B}(p)_*$ is equivalent to $\mathcal{A}(p)_*$, we now show that $\mathcal{B}(p)_*$ and $\mathcal{A}(p)_*$ are equivalent to the categories of $E(1)_*E(1)$-comodules and $K_{(p)*}K_{(p)}$-comodules. Thus our main results may be reformulated using these comodule categories. We refer the reader to [2], [5], or [13] for an introduction to the relevant theory of coalgebras and comodules.

We begin with preliminaries leading to the equivalence of $\mathcal{B}(p)_*$ with the category of $E(1)_*E(1)$-comodules. Recall that an object $M \in \mathcal{B}(p)_*$ automatically has a left module structure over $\pi_*E(1) = Z_{(p)}[v, v^{-1}]$. By a *$\pi_*E(1)$-bimodule in $\mathcal{B}(p)_*$* we mean an object $M \in \mathcal{B}(p)_*$ with a right $\pi_*E(1)$-module structure such that $(xm)y = x(my)$ and $\psi^k(my) = (\psi^k m)y$ for each $x, y \in \pi_*E(1)$, $m \in M$, and $k \in Z_{(p)}^*$.

LEMMA 10.1. *If B is a $\pi_*E(1)$-bimodule in $\mathcal{B}(p)_*$ and G is a left $\pi_*E(1)$-module, then $B \otimes_{\pi_*E(1)} G$ is in $\mathcal{B}(p)_*$.*

Proof. For $x \in \pi_*E(1)$, $b \in B$, $g \in G$, and $k \in Z_{(p)}^*$, we let $x(b \otimes g) = xb \otimes g$ and $\psi^k(b \otimes g) = \psi^k b \otimes g$. Now form a short exact sequence $0 \to F' \to F \to G \to 0$ of left $\pi_*E(1)$-modules with F' and F free, and note that $B \otimes_{\pi_*E(1)} F'$ and $B \otimes_{\pi_*E(1)} F$ are in $\mathcal{B}(p)_*$. Thus the cokernel $B \otimes_{\pi_*E(1)} G$ is also in $\mathcal{B}(p)_*$.

By 4.2, $E(1)_*E(1)$ is a $\pi_*E(1)$-bimodule in $\mathcal{B}(p)_*$, and we let $\epsilon : E(1)_*E(1) \to \pi_*E(1)$ be the map induced by the multiplication $\mu : E(1) \wedge E(1) \to E(1)$. For a left $\pi_*E(1)$-module G, consider the diagram

$$\begin{tikzcd} E(1)_*E(1) \otimes_{\pi_*E(1)} G \arrow[r, "\bar{\epsilon}"] \arrow[dr, "\epsilon"'] & \mathcal{U}(G) \arrow[d, "e"] \\ & G \end{tikzcd}$$

where ϵ is given by $\epsilon(x \otimes g) = \epsilon(x)g$ for $x \in E(1)_*E(1)$ and $g \in G$. By 6.5 there is a unique map $\bar{\epsilon}$ in $\mathcal{B}(p)_*$ making the diagram commute.

LEMMA 10.2. *The map $\bar{\epsilon} : E(1)_*E(1) \otimes_{\pi_*E(1)} G \to \mathcal{U}(G)$ is an isomorphism for each G. Moreover, for $M \in \mathcal{B}(p)_*$ each left $\pi_*E(1)$-module map $f : M \to G$ has a unique lifting $\bar{f} : M \to E(1)_*E(1) \otimes_{\pi_*E(1)} G$ in $\mathcal{B}(p)_*$ with $f = \epsilon\bar{f}$.*

Proof. If G is a free left $\pi_*E(1)$-module on one generator, then $\bar{\epsilon}$ is an isomorphism by 6.6. The general isomorphism now follows since $\mathcal{U}$ preserves direct sums and is exact. The last statement follows from the universal property of $\mathcal{U}$.

10.3. *The equivalence of $\mathcal{B}(p)_*$ with the category of $E(1)_*E(1)$-comodules.* For $M \in \mathcal{B}(p)_*$ let $\Delta : M \to E(1)_*E(1) \otimes_{\pi_*E(1)} M$ be the unique map, given by 10.2, in $\mathcal{B}(p)_*$ such that $\epsilon\Delta = 1: M \to M$. This gives M the structure of an $E(1)_*E(1)$-comodule by an easy formal argument using 10.2. For an $E(1)_*E(1)$-comodule N with comultiplication $\Delta: N \to E(1)_*E(1) \otimes_{\pi_* E(1)} N$, the bar resolution gives an exact sequence

$$0 \to N \xrightarrow{\Delta} E(1)_*E(1) \otimes_{\pi_* E(1)} N \xrightarrow{\Delta\otimes 1 - 1\otimes\Delta} E(1)_*E(1) \otimes_{\pi_* E(1)} E(1)_*E(1) \otimes_{\pi_* E(1)} N.$$

Since $\Delta \otimes 1 - 1 \otimes \Delta$ is clearly in $\mathcal{B}(p)_*$, its kernel N has a canonical structure in $\mathcal{B}(p)_*$. The foregoing constructions are inverse to each other and provide additive equivalences between $\mathcal{B}(p)_*$ and the category $E(1)_*E(1)$-comodules. A similar equivalence involving torsion $E(1)_*E(1)$-comodules was obtained by Ravenel [11] using very different methods. For $X \in \underline{Ho}^s$ the structure of $E(1)_*X$ in $\mathcal{B}(p)_*$ corresponds via the above constructions to its standard structure as an $E(1)_*E(1)$-comodule. Thus $\mathcal{B}(p)_*$ may be replaced by the category of $E(1)_*E(1)$-comodules in our main results.

To show the equivalence of $\mathcal{A}(p)_*$ with the category of $K_{(p)*}K_{(p)}$-comodules, we recall that an object $M \in \mathcal{A}(p)_*$ automatically has a left module structure over $\pi_* K_{(p)} = Z_{(p)}[u, u^{-1}]$. By a *$\pi_* K_{(p)}$-bimodule in $\mathcal{A}(p)_*$* we mean an object $M \in \mathcal{A}(p)_*$ with a right $\pi_* K_{(p)}$-module structure such that $(xm)y = x(my)$ and $\psi^k(my) = (\psi^k m)y$ for each $x, y \in \pi_* K_{(p)}$, $m \in M$, and $k \in Z^*_{(p)}$. If A is a $\pi_* K_{(p)}$-bimodule in $\mathcal{A}(p)_*$ and G is a left $\pi_* K_{(p)}$-module, then $A \otimes_{\pi_* K_{(p)}} G$ is in $\mathcal{A}(p)_*$ by the argument of 10.1. By 2.2, $K_{(p)*}K_{(p)}$ is a $\pi_* K_{(p)}$-bimodule in $\mathcal{A}(p)_*$, and we let $\epsilon: K_{(p)*}K_{(p)} \to \pi_* K_{(p)}$ be the map induced by the multiplication $\mu: K_{(p)} \wedge K_{(p)} \to K_{(p)}$. For a left $\pi_* K_{(p)}$-module G let $\epsilon: K_{(p)*}K_{(p)} \otimes_{\pi_* K_{(p)}} G \to G$ be given by $\epsilon(x \otimes g) = \epsilon(x)g$ for $x \in K_{(p)*}K_{(p)}$ and $g \in G$.

LEMMA 10.4. *For a left $\pi_* K_{(p)}$-module G and for $M \in \mathcal{A}(p)_*$, each left $\pi_* K_{(p)}$-module map $f: M \to G$ has a unique lifting $\bar{f}: M \to K_{(p)*}K_{(p)} \otimes_{\pi_* K_{(p)}} G$ in $\mathcal{A}(p)_*$ with $f = \epsilon\bar{f}$.*

Proof. By the equivalence of $\mathcal{A}(p)_*$ and $\mathcal{B}(p)_*$ in 3.8, it suffices to show that each left $\pi_* E(1)$-module map $f: M^{[0]} \to G$ has a unique lifting $\bar{f}: M^{[0]} \to E(1)_*K_{(p)} \otimes_{\pi_* K_{(p)}} G$ in $\mathcal{B}(p)_*$ with $f = \epsilon\bar{f}$. For this it suffices to show that $\bar{\epsilon}: E(1)_*K_{(p)} \otimes_{\pi_* K_{(p)}} G \to \mathcal{U}(G)$ is an isomorphism in $\mathcal{B}(p)_*$ for each $\pi_* K_{(p)}$-module G. If G is a free $\pi_* K_{(p)}$-module on one generator, then $\bar{\epsilon}$ is an isomorphism by 6.6. The general case follows since $\mathcal{U}$ preserves direct limits and is exact.

10.5 *The equivalence of $\mathcal{A}(p)_*$ with the category of $K_{(p)*}K_{(p)}$-comodules*. In view of 10.4, the constructions in 10.3 apply to show that each $M \in \mathcal{A}(p)_*$ has a canonical $K_{(p)*}K_{(p)}$-commodule structure and that each $K_{(p)*}K_{(p)}$-commodule N has a canonical structure in $\mathcal{A}(p)_*$. The constructions are inverse to each other and provide additive equivalences between $\mathcal{A}(p)_*$ and the category of $K_{(p)*}K_{(p)}$-comodules. For $X \in \underline{Ho}^s$ the structure of $K_{(p)*}X$ in $\mathcal{A}(p)_*$ corresponds via the above constructions to its structure as a $K_{(p)*}K_{(p)}$-comodule. Thus $\mathcal{A}(p)_*$ may be replaced by the category of $K_{(p)*}K_{(p)}$-comodules in our main results over $K_{(p)*}$ (see 7.10, 8.12, 9.11).

10.6. *The equivalence of the category of $E(1)_*E(1)$-comodules with that of $K_{(p)*}K_{(p)}$-comodules.* In 3.8 we established an additive equivalence from $\mathcal{B}(p)_*$ to $\mathcal{A}(p)_*$ sending $M \in \mathcal{B}(p)_*$ to $\pi_*K_{(p)} \otimes_{\pi_*E(1)} M \in \mathcal{A}(p)_*$. This corresponds via 10.3 and 10.5 to a (presumably well-known) additive equivalence from the category of $E(1)_*E(1)$-comodules to that of $K_{(p)*}K_{(p)}$-comodules, sending an $E(1)_*E(1)$-comodule N to the $K_{(p)*}K_{(p)}$-comodule $\pi_*K_{(p)} \otimes_{\pi_*E(1)} N$ with comodule structure induced via the isomorphism

$$K_{(p)*}K_{(p)} \approx \pi_*K_{(p)} \otimes_{\pi_*E(1)} E(1)_*E(1) \otimes_{\pi_*E(1)} \pi_*K_{(p)}.$$

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