HILL REDUCTION THEOREM

Thm \[ MU^G A \leq S^0 = H \exists \]

\[ R(\infty) \]

Pf: By induction on \([G]\)

For \(|G| = 1\), start is due to Quillen (?)

\[
\begin{array}{ccc}
EP_{+} \times R(\infty) & \longrightarrow & R(\infty) \\
\downarrow & & \downarrow \\
EP_{+} \times H_Z & \longrightarrow & EP_{+} \times H_Z \\
\end{array}
\]

\(P = \text{family of fibers}\)
To show $R$ is again

1. Compute $\Pi_x \bar{G} \cdot R(\infty)$ and $\Pi_x \bar{G} \cdot H_2$

are abstractly isomorphic

2. Show $R_x$ induces the isomorphism

To show $L$ is an again

\[
\begin{align*}
\text{Show } & i_x^* \left[ R(\infty) \right] \rightarrow i_x^* \left[ H_2 \right] = H_2 \\
R_y(\infty) \qquad R_{x_1}(\infty) = MV((G)) \wedge S^3
\end{align*}
\]

\[
A = N_{C_2}^G \left( S^0 \left[ \bar{H}_1, \bar{H}_2, \ldots \right] \right)
\]

$\triangledown \in \text{Sp} \left( \mathbb{R}^2 \right)$
The map \( A \to \text{MV}(\mathbb{C}) \) is obtained by using ring structure of target

\[
\Phi^G_{\text{MF}}(\mathbb{C}) = \Phi^G_{\text{MF}}(\text{MV}(\mathbb{C})) \land \Phi^G_{\text{MF}}A
\]

and

\[
\Phi^G_{\text{MF}} \text{ MV}(\mathbb{C}) = \mathcal{D}
\]

\[
\Phi^G_{\text{MF}}(A) = \Phi^G_{\text{MF}} \left( \mathcal{S} \left[ \mathcal{S}N_{r_1}, N_{r_2}, \ldots \right] \right)
\]

\[
= \mathcal{S} \left[ \Phi^G_{\text{MF}}N_{r_1}, \Phi^G_{\text{MF}}N_{r_2}, \ldots \right]
\]

\[
= \mathcal{S} \left[ \mathcal{S} \left[ \Phi^G_{\text{MF}}N_{r_1}, \Phi^G_{\text{MF}}N_{r_2}, \ldots \right] \right] = \mathcal{B}
\]

\[
\Phi^G_{\text{MF}}(\mathcal{S}) = \mathcal{S}
\]
\[ x_0 \, g^* \, R_0 (x) = M_0 \wedge S^0 \]

\[ = H\hat{F}_2 \left( h_1, h_2, \ldots, h_i \mid i \neq 2^{\nu_1} \right) \wedge S^0 \]

\[ = H\hat{F}_2 \wedge S^0 \]

\[ S^0 \left\{ b_1, b_2, \ldots \right\} \]

Can translate this into iterated cofibers, and we get

\[ H_2 \, g^* \, R (x) \begin{cases} \hat{F}_2 & \text{if } x = 0 \text{ and } x \text{ even} \\ 0 & \text{else} \end{cases} \]

Can also treat this as a Kunneth problem.

There is no multiplicative structure here.
\[ E_i^2 HZ = (S_i^0 \cap HZ)^G \]

\[ S_i^0 HZ \rightarrow S_i^0 HZ \rightarrow S_i^0 HZ \]

This leads to a chain \( \cdots \)

\[ 2 \xleftarrow{\text{inj}} \mathbb{Z}[G] \xrightarrow{\text{rk}} 2 \mathbb{Z}[G_2] \xrightarrow{\text{rk}} 2 \mathbb{Z}[G_2] \xleftarrow{\text{inj}} \cdots \]

Taking fixed pts gives \( \cdots \)

\[ 2 \xleftarrow{\text{inj}} \mathbb{Z} \xrightarrow{\text{rk}} \mathbb{Z} \xrightarrow{\text{rk}} \mathbb{Z} \xleftarrow{\text{inj}} \cdots \]

\[ \prod_i E_i^0 HZ = \begin{cases} 2^{1/2} & \text{for } i=0 \text{ and } i \text{ even} \\ 0 & \text{else} \end{cases} \]

Thus, \( H \cap R_0(\infty) \) and \( \prod H \cap R_0(\infty) \) are ...
Let \( R = i^*_H \text{MU}^{(G)} \) and \( H = \text{subgp of index 2} \).

By induction, \( \text{MU}^{(H)} \) has the expected slices

\[
R = \text{MU}^{(H)} \cap \text{MU}^{(H)}
\]

\[
= \text{MU}^{(H)} \left[ \left. H \times X_{M_1^G}^{G_1} \cap \cdots \right| H \times X_{M_2^G}^{G_2} \cap \cdots \right]
\]

\[
i^*_H \text{R}_{\alpha^0}(\infty) = i^*_H \left( \text{MU}^{(G)} \cap S^0 \right) \text{A}
\]

\[
= i^*_H \left( \text{MU}^{(G)} \cap \text{MU}^{(G)} \right) \text{A}
\]

\[
= R \cap i^*_H A \left( R \right.
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\[ R \cong \mathbb{V}_H \cong R \left[ \mathbb{H}_0 \cong \mathbb{G}_0, \ldots \right] \]

\[ = R \left[ \mathbb{H}_0 \cong \mathbb{G}_0, \ldots ; \mathbb{H}_2 \cong \mathbb{G}_2, \ldots \right] \]

\[ R_H(\infty) = \operatorname{MU}^{\langle H \rangle} \wedge \mathbb{A}^1_{\mathbb{H}} \cong \operatorname{MU}^{\langle H \rangle} \wedge \mathbb{A}^1_{\mathbb{H}} \cong R \left[ \mathbb{H}_0 \cong \mathbb{G}_0, \ldots ; \mathbb{H}_2 \cong \mathbb{G}_2, \ldots \right] \]

\[ \operatorname{MU}^{\langle H \rangle} \wedge \mathbb{A}^1_{\mathbb{H}} = \operatorname{MU}^{\langle H \rangle} \left[ \mathbb{H}_0 \cong \mathbb{G}_0, \ldots ; \mathbb{H}_2 \cong \mathbb{G}_2, \ldots \right] \]

\[ = R \left[ \mathbb{R} \mathbb{H}_0 \cong \mathbb{G}_0, \ldots ; \mathbb{H}_2 \cong \mathbb{G}_2, \ldots \right] \]

\[ \text{base change along } \operatorname{MU}^{\langle H \rangle} \rightarrow R \]

Compare 1 and 2
Thus it suffices to show
\[ R \left[ H \bar{\eta}_1^H, \ldots, H \bar{\eta}_1 \bar{\eta}_2 \bar{\eta}_3, \ldots \right] = R \left[ H \bar{\eta}_1 \bar{\eta}_2, \ldots, H \bar{\eta}_1 \bar{\eta}_2 \bar{\eta}_3, \ldots \right] \]

Let \( I \) be the ideal generated by
\( (H \bar{\eta}_1 \bar{\eta}_2, \ldots, H \bar{\eta}_1 \bar{\eta}_2 \bar{\eta}_3, \ldots) \)

\[ \bar{\eta}_i^H = \sum \bar{\eta}_i^G + \lambda \bar{\eta}_i^G \quad \text{if } i \neq 2 + 1 \quad \text{or vice versa} \]

This follows from def of \( \bar{\eta} \)

Thus \( \bar{J}^0 = (H \bar{\eta}_1^H, \ldots, H \bar{\eta}_1 \bar{\eta}_2 \bar{\eta}_3, \ldots) \) is \( I \).
Goal: Build a map of associative rings
(i.e. $A_2$-rings) such that
$\pi^H_n \to$ polynomial in $\tilde{\pi}_n^G \times \tilde{\pi}_n^G$
expressing it.

Assume we can:

$R[H, \pi_n^H, \ldots, H \times \pi_n^G, \ldots] \to R[H, \tilde{\pi}_n^G, \ldots, H \times \tilde{\pi}_n^G, \ldots]$

and it's an underlying equivalence.

Slices of $MU_{(4)}$ are what we expect:

* odd slices vanish
* even slices (isotropic regular cells) $\to HZ$
Lemma 1) R[ – J also has these properties.

2) \( f: X \rightarrow Y \) with \( X, Y \) have these properties and underlying map is an equiv., then \( f \) is an equivalence.

This says that under these hypotheses, an ordinary equiv. is a \( \mathcal{L} \)-equiv.
2) By considering the slice tower, it suffices to show this for
\[ X = ( \text{single regular cell} ) \cap H \]
\[ Y = ( \text{single regular cell} ) \cap H \leq \]

Apply isotropy separation to reduce to case of no induced cells and check it.

**EXERCISE**

Def: An **A∞**-ring is weakly commutative if it is an **A∞**-retract of an **E∞**-ring.

\[ A \to E \to A \]

The rings \( R[-J] \) above are
Weakening comm. Weak commutativity is good enough to construct the maps we need.

\(R[\cdot - \cdot]\) is a retract of \(R \cdot R\)