Thm (HHR) If $M$ is a smooth framed manifold with Kervaire invariant 1, then $\dim M$ is $2, 6, 14, 30, 62, 126$.

This solves a long standing problem.

1930s: Understood degree of $\mathbb{R}^n \to S^n$ homology + cohomology.

Pontryagin studied maps $S^m \to S^n$ inverse image of regular value is a framed $M^k$. Preimage of closed
path in a cobordism between two such manifolds.

Thus we get

cobordism classes of stably framed $\mathbb{R}$-manifolds

\[ 
\Psi_n(\mathbb{R}) \cong S^n \quad \text{for } n \gg 0
\]

Pontrjagin used this to study $\Psi_n(\mathbb{R})$ for small $n$.

\[ 
\Psi_n S^n = 2 \quad \text{(degree of a map)}
\]

\[ 
\Psi_{n+1} S^n = 2/2 \quad \text{(two framings of } S^1 \text{)}
\]

\[ 
\Psi_{n+2} S^0 = 0 \quad \text{by mistake}
\]

studied framed surfaces and
framed surgery. There is an obstruction having to do with framings on closed curves leading to \( \eta: H^4(M) \to \mathbb{Z}/2 \)

\( \eta \) is not linear. \( \eta(x+y)-\eta(x)-\eta(y) = \langle xy \rangle \)

\( \text{Arf}(\eta) \in \mathbb{Z}/2 \) and \( \text{Th}_{n+2}(S^n) = \mathbb{Z}/2 \)

\( M \to \text{Arf}(\eta) \)

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Q: When is every framed manifold cobordant to a "sphere"? i.e., a topological sphere.

A: Always except in the dimensions of the theorem.
Late 1950s + early 60's work of Kervaire–Milnor showed that is vanishes for \( k = 0 \).

Kervaire's hand cuffs

\[ \nabla^0 = \mathcal{X}_g. \quad \text{hence to } S \]

\[ m^0 = \nabla^0 \cup CX_g \]

This \( M \) has no smooth structure.
Could replace $k$ by $2k+1$ giving $N^{4k+2}
\text{homeo to } S^{4k+1}$
$M^{4k+2}$ PL manifold

Q is $X_{4k+1}$ diffeo to $S^{4k+1}$?

Q is $M$ smoothable?

Answer: Almost never.
KM could not answer these equivalent questions.
Q: In which dimension does there exist a smooth, stably framed, nilpotent Kervaire invariant 1? This is answered by our theorem.

Broadley 1969. $\Phi(M) = 5$ exists with $\Phi(M) = 5$

$\Rightarrow \exists \theta \in \pi_*^G$ represented by $h_j$ in

the Adams spectral sequence

$k = 2^i - 2$. dim $M$ must have this form.

Barratt, Jones, Mahowald + Tangora showed $\exists \theta_j$ for $j \leq 5$ by 1984

$\theta_1, \theta_2$ and $\theta_3$ were known before.
Many believed that all $g_i$ exist. Now that we know there are only $3$ or $6$ of them, we look for constructions related to exceptional Lie groups

**EHP sequence**

$$
\pi_k S^1 \to \pi_{k+1} S^1 \to \pi_{k+1} S^{2m+1} \to \pi_{k+1} S^m
$$

This leads to an inductive process starting with $\pi_4 S^1$.

Suppose $k = 2n$. So $\pi_{k+1} S^{2m+1} = \mathbb{Z}$. Generate maps to $\mathbb{Z}_{2m+1} S^n$, the Whitehead square
Q1. For which $g$ is $[L_m, L_n]$ in image of $E^0$?

Q2. Is $[L_m, L_n]$ divisible by 2?

Q1. $[L_m, L_n]$ in image $E^0 \rightarrow S^m$ has $j$ linearly independent vector fields [JAMS].

Solved by Adams 1962.

Q2. For even $n$ this is the Hopf invariant one problem, which was also solved by Adams. For $n$ odd it is the Kervaire invariant question.
The proof. We introduce a colimn theory $S_I$ with

1. Detachment Theorem $S_I$ if $F$, it has a nontrivial image in $T_x S_I$.
2. Periodicity Theorem $S_{k+256} = S_k$.
3. Map Theorem $S_{-k} = 0$ for $-1 < k < 0$. 
Informal discussion

Want to prove reducibility claim

**Theorem (Conclusion)** Let $M^n$ be a "real" manifold with $M^2 \cong N^n$. If $N^n$ is an unoriented boundary then $M^n$ cobordant to a free $C^\infty$-manifold.

For each an $M^{2n}$ we have a double cover

$$M^{2n} \rightarrow M^{2n}/C_2.$$ Then $S_{\overline{M}}$.
Example \( \mathbb{P}^1 \) fixed at \( \mathbb{RP}^1 \)

\[ x^2 + y^2 = z^2 \]  quadratic for \( z \in \mathbb{C} \)

For \( z = 1 \) it is real and \( \tilde{\mathbb{C}} \). For \( z = -1 \), there are no real points, so the \( \mathbb{C} \)-action is free. The orbit space is \( \mathbb{RP}^2 \), where \( m^2 \neq 0 \).

\[ T_x \mathbb{C}^2 = M_{\mathbb{R}} \mathbb{C}^2 \]

\[ T_x \mathbb{C}^2 \mathbb{H}^2 = \frac{2}{\sqrt{2}} [a] \quad |a| = |b| = 2 \]

(not obviously arising)

\[ T_x \mathbb{C}^2 \mathbb{H}^2 = \frac{2}{\sqrt{2}} [b] \]
The map above is related to characteristic numbers $w_j^{2n}$ as in the example above for $n = 1$.

The $C_4$ and

$\text{MU}^{(2)} = \text{MU}_1 \times \text{MU}_0 \quad \text{has } C_4 \text{-action}$

$\pi_4 \text{MU}^{(2)} = \mathbb{Z} \left[ x_1, y x_1, x_2, y x_2, \ldots \right]$  

$\text{MU}^{(2)}/(x_1, y x_1, \ldots) \cong HZ$

as before this can be reduced to geometric fixed points
\[ M \xrightarrow{\nu} BU \times BU \xrightarrow{C_4 \text{- map}} \]

\[ (N, W) \rightarrow (W, \bar{V}) \]

\( V \oplus \mathbb{R} \times V \) is stable normal bundle

\( M \) has \( C_4 \) action restricting to a real structure over \( C_2 \), with involution \( J \).

**Example** Let \( X \) be a real manifold. If it is a complex bundle \( V \) which is its stable normal bundle \( (\ast \times X, p^* V) \) is a \( C_4 \)-mfld as above.
Thus given a $C_4$-model $M$, $M^{C_2}$ is unoriented. If $M^{C_4} = \emptyset$ then $M$ is unoriented to $M^{'\prime}$ with $(M^{'\prime})^{C_4} = \emptyset$.

This is related to $(E_{C_2} \to S^0 \to \widetilde{E_{C_2}}) \wedge MU^{C_2}$.

Assume $M^{C_4} = \emptyset$. $M^{C_2}$ has a free action of $C_2' = C_4/C_2$. We can work as before and use

$$\int_{M^{C_2}/C_2'} \dim$$
Example \((\mathbb{CP}^1 \times \mathbb{CP}^1)^\mathbb{Z}_2 = \{(a, a) : (a, a) \sim (b, a)\}\rangle = \mathbb{RP}^1\)

Let \(\mathbb{CP}^1\) be defined by \(x^2 + y^2 = z^2\) (different complex structure)

A real cobordism is a map \(N \to \mathbb{R}\) transverse at 0 and 1
A complex \(N \to \mathbb{C}\)

\(\text{equiv. for } \mathbb{C}_2\)-actions on \(N\) and \(\mathbb{C}\)

E.g., \(\{(x, y, z) \in \mathbb{CP}^2 : x^2 + y^2 - z^2 = 0\}\) is complex cobordant to \(\{(x, y, z) \in \mathbb{CP}^2 : x^2 + y^2 + z^2 = 0\}\)