In 100 people present.
Joint with Hill + me.

Thm (HHR) If $M$ is a smooth stably framed manifold with Kervaire invariant one, then $\dim M = 2, 6, 14, 30, 62, 126$. 
History of problem

1930s. Homology + degree had been defined.

Portraying in study $S^n \times \mathbb{R} \to \mathbb{R}$ in terms of $f'(x)$ for a regular value $x$.

It is a smooth mfd of dim $k$ with a framing of its normal bundle in $S^n \times \mathbb{R}$.

Choosing a different regular $x_j$ leads to a regular mfd $M_j$ which is framed cobordant to $M_0$. 

There is a bijection

\[ T^m \times R \leftrightarrow \text{cobordism of } \text{framed } \mathbb{R}^m - \text{mfds in } \mathbb{R}^m \times R \]

He computed this up for small \( k \).

For \( k = 0 \) we get \( T^m S^m = \mathbb{Z} \).

For \( k = 1 \), \( S^1 \) has two framings

\[ T^m S^m = \mathbb{Z}/2 \]
For $k=2$, $S^2$ has a framing that extends to a disk. Any other framing differs by an element in $\pi_2 \cong \mathbb{Z}$. Hence it represents the trivial class in $\pi_{n+2} S^n$.

Suppose $M$ has genus $1$.

Cut along red arc and attach disks. The obstruction is an element.
in \( \pi_1(S^1 \times S^1) \) = \( \mathbb{Z}/2 \mathbb{Z} \). Thus we get a map \( H_1(M; \mathbb{Z}/2) \to \mathbb{Z}/2 \).

If \( \phi \) is linear then ker \( \phi \) has nontrivial element and \( \pi_1(M; \mathbb{Z}/2) \) = 0.

Red and blue arcs have trivial framing but the green arc (their sum) does not.

Hence \( \phi \) is nonlinear.
This leads to the definition of $\text{Aff}(\mathbb{R})$.

Question: In which dimensions is every framed mfd cobordant to a sphere? Answer is given by theorem. Pontryagin's method would work in all dimensions except 2, 6, 14, 30, 62, and 126.
Milnor 1956 constructed an M homomorphism but not diffeomorphism to $S^7$.

Kervaire-Milnor (56, 63) $H^1 = gp \text{ of } M^k$ homeo to $S^k$ under connective sum.

(up to $h$-cobordism). The computed it in terms of $Th(n, S^n)$ up to a factor of 2. Our theorem settles the factor 2 (except in dim 126).
Kervaire 1960

Let $M^{2n}$ be a framed $(n-1)$-connected $(2n)$-mfd. He constructs $\Phi : H_n(M; \mathbb{Z}) \to \mathbb{Z}$

$\Phi(M)^\circ = \text{Arf}(\Phi) = \text{Kervaire invariant}$

He constructed a PL mfd $M$ in dim 10 with $\Phi(M) = 1$. Showed any smooth $10$-mfd has $\Phi(M) = 0$.

Hence this $M$ is nonsmoothable.

Question: In which dim can $\Phi(M)$ be nonzero? Answered by Theorem.
Browder 1969. If $\mathcal{O}(M) = 1$, then $\dim M = 2^{j+1}$.

In this dimension an M exists $\Rightarrow$

$\exists \beta \in \prod_{2^j+1} (S^2)$ representing $\beta^2$ via classical Adams 55.

This problem remained open for 40 years.

Theorem (Barrett- Jiang- Mahowald) $\exists \beta_j$ for $j \leq 5$. [$j=6$ is still open].

Is there a connection to the six exceptional Lie groups? ??
EHP sequence (James, ~1958)

\[ \cdots \to \Omega m(S^n) \overset{\text{localize}}{\to} \Omega m+1(S^{n+1}) \overset{\text{H}}{\to} \Omega m+1(S^{n+1}) \overset{P}{\to} \Omega m(S^n) \to \cdots \]

It leads to an inductive process.

If \( m = 2n \) we have

\[ \Omega_{2n+1}(S^{2n+1}) \overset{\text{Whitehead product}}{\to} \Omega_{2n+1}(S^{2n}) \]

**Questions**

Q1: How far does \( \Omega m \) descend?

Q2: Do \( Q_m \) divisible by 27?
Q1 is equivalent to vector field problem: \( w \) deforms \( k \) times \( S^n \) has \( k \) vector fields.

Q2: \( n \) even: Hopf invariant one problem
\( n \) odd: Kervaire invariant problem

**Sketch of proof**

Construct a spectrum \( \hat{S}^2 \) with \( C_8 \)-action.
\[
\hat{S}^2 = D^{-1} M \mathbb{R}^{(2)}
\]
\[
S^2 = \hat{S}^2 \in C_8 \quad \text{(anti equality)}
\]
S2 has 3 properties:

**Detection Thm**: If \( f \neq 0 \), it's image in \( \Pi \times S2 \) is nonzero.

**Periodicity Thm**: \( \Pi\times S2 \cong \Pi \times S2 \) \( (52^4) \)

**Gap Thm**: \( \Pi_k S2 = 0 \) for \( -4 < k < 0 \) \( (52^4) \)

PT has to do with equivariant methods.

The use of equivariant theory is essential. It was developed by May.