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Note Title

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Abstract alg. geo

I Origin of schemes

II Schemes as spaces

III Sheaves

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V Sheaves of modules + quasi coherence

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I. Origins

Classical AG over alg closed field k

affine varieties e.g.

$V = Z(y - x^2) \subseteq \mathbb{C}^2 \longleftrightarrow A(V) =$ coord ring $= \mathbb{C}[x,y]/(y-x^2)$

zero locus

of polynomial

Abstract AG

$\mathrm{Spec}(\mathbb{Z}) \longleftrightarrow$ arbitrary ring e.g. $\mathbb{Z}$

affine scheme

II Schemes as spaces. R = comm ring with 1

Def $\mathrm{Spec}(R) = \left\{ \begin{array}{c} \text{proper} \\ \text{prime ideals} \\ \text{of } R \end{array} \right\}$ as set

Example $\mathrm{Spec}(\mathbb{Z}) = \{ (2), (3), (5), \ldots, (0) \}$

Ex $\mathrm{Spec}\, \mathbb{C}[X, Y] \ni \underline{m} = (X - z, Y - w)$

= max ideal corresponding to $(z, w) \in \mathbb{C}^2$

Another prime ideal is $p = (Y - X^2)$ which corresponds to a curve

At the level of sets, a scheme X is

$X = \coprod_{i \in I} \operatorname{Spec}(R_i) \Big/ \begin{matrix}\text{gluing} \\ \text{construction}\end{matrix}$

(see Eisenbud + Harris, p. 33 "Geometry of Schemes")

Topology of Spec (A)

Let $f \in A$. The basic open set

$V(f) = \{ p \in \operatorname{Spec} A : f \notin p \}$

e.g. $\operatorname{Spec}(\mathbb{Z})$ $\quad V(f) = \{ \text{primes not dividing } f \}$

$=$ compliment of finite set

$= \{ \mathfrak{p} \mid \text{"} f(\mathfrak{p}) \neq 0 \text{"} \}$

For this we need to view f as a function

$\operatorname{Spec} \mathbb{Z} \xrightarrow{f}$ "various fields" = product of all residue fields

(more generally the quotient field of $A/\mathfrak{p}$)

e.g. $f = 28$ $p = 17$

$$\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/(17)$$

$f((p)) = 28((17)) = \pi(28) =$ mod 17 reduction of 28

We are reducing 28 modulo every prime.

<u>Def</u> The regular functions on $\operatorname{Spec} \mathbb{Z}$ are just the integers $f \in \mathbb{Z}$.

$V(f) = \operatorname{Spec} \mathbb{Z}[\frac{1}{f}]$

① localization
② ring of regular functions on $V(f)$.

The topology here is very weak.

III Sheaves

Example $M = C^\infty$ mfd

U = open set

$C^\infty(U)$ = ring of C^∞ fns on U

We get a contravariant functor $\mathcal{F} : \left\{\begin{matrix}\text{open sets} \\ \text{of } M\end{matrix}\right\} \longrightarrow$ Rings

This is a sheaf of rings over M.

<u>Def</u> A sheaf on a space X is a cont. functor

$\mathcal{F} : \begin{Bmatrix} \text{open} \\ \text{sets} \end{Bmatrix} \longrightarrow$ target category

with some gluing conditions

Spec $\mathbb{Z}$ has a sheaf $\mathcal{O}$ of rings on it

For $f \in \mathbb{Z}$, $\mathcal{O}(U(f)) = \mathbb{Z}[\frac{1}{f}]$. This is the <u>structure sheaf</u> of Spec$(\mathbb{Z})$. We say Spec$(\mathbb{Z})$ is a ringed space, i.e. a space with a sheaf of rings.

Stalks e.g. M = mfd as before

$p \in M$. Let f, g be C^∞ fns defined on a nbd of p.

$f \sim g$ if $f = g$ on some nbd of p

Def $\left\{ \begin{array}{l} C^\infty \text{ fns defined} \\ \text{near } p \end{array} \right\} / \sim$ = set of germs @ p.

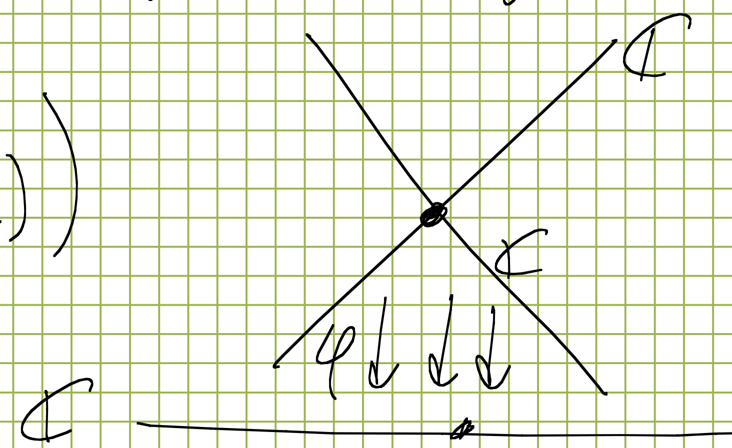
Stalks on $(\operatorname{Spec} \mathbb{Z}, \mathcal{O})$

For $(p) \in \operatorname{Spec} \mathbb{Z}$, the stalk @ p is

$$\mathcal{O}_{(p)} = \varinjlim_{p \in U} \mathcal{O}(U) \cdots = \mathbb{Z}_{(p)}$$

This makes $\operatorname{Spec}(\mathbb{Z})$ a locally ringed space, i.e. it has a sheaf of rings with local stalks.

<u>Ex</u>: $X = \operatorname{Spec}\left(\mathbb{C}[x,y]/(xy)\right)$



$Y = \operatorname{Spec} \mathbb{C}[t]$

<u>Exercise</u>: Show the map $\mathbb{C}[t] \xrightarrow{\varphi} \mathbb{C}[x,y]/(xy)$

$t \longmapsto x+y$

induces

V Sheaves of modules. Let $(X, \mathcal{O}_X)$ be a scheme.

Rough def A sheaf $\mathcal{F}$ on X is a sheaf of $\mathcal{O}_X$ modules if $\forall$ open $U \subset X$, $\mathcal{F}(U)$ is a module over $\mathcal{O}_X(U)$ with some technical conditions.

Let A = ring, M = A-module $\mathcal{O}_A$-

Can use M to build a sheaf of modules $\tilde{M}$ (see Hartshorne, p. 110)

Def. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on a scheme X is quasi-coherent if

$$X = \bigcup_i \operatorname{Spec}(A_i) / \sim$$

and for each i, $\exists$ an A_i-module M_i s.t. $\mathcal{F}|_{U_i} \simeq \widetilde{M_i}$.

Such sheaves have many nice properties.