On the Non-Existence of Kervaire Invariant One Manifolds

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Main Result

Theorem (H.-Hopkins-Ravenel)

There are smooth Kervaire invariant one manifolds only in dimensions 2, 6, 14, 30, 62, and maybe 126.
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Exemplars:
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Exemplars:

2. $S^1 \times S^1$
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Exemplars:

- $S^1 \times S^1$
- $SU(2) \times SU(2)$
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- $S(O) \times S(O)$
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Exemplars:

- $S^1 \times S^1$
- $SU(2) \times SU(2)$
- $S(\mathbb{O}) \times S(\mathbb{O})$
- (Bökstedt) Related to $E_6/(U(1) \times \text{Spin}(10))$
- Possibly a similar construction.
1930s Pontryagin proves
\[ \{ \text{framed } n - \text{ manifolds} \} / \text{cobordism} \cong \pi_n^S. \]
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1950s Kervaire-Milnor show can always reduce to case of spheres
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Tries to use surgery to reduce to spheres & misses an obstruction.

1950s  Kervaire-Milnor show can always reduce to case of spheres
Except possibly in dimension \( 4k + 2 \), where there is an obstruction: Kervaire Invariant.
Adams Spectral Sequence

\[ [X, Y] \]
Adams Spectral Sequence

\[ \left[ X, Y \right] \longrightarrow \text{Hom}_A(H^*(Y), H^*(X)) \]
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\[ [X, Y] \xrightarrow{\sim} \text{Hom}_A(H^*(Y), H^*(X)) \]

Have a SS with

\[ E_2 = \text{Ext}_A(H^*(Y), H^*(X)) \]

and converging to \([X, Y]\).
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(Adem) \(\text{Ext}^1(F_2, F_2)\) is generated by classes \(h_i, i \geq 0\).
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and converging to \([X, Y]\).

- (Adem) \( \text{Ext}^1(F_2, F_2) \) is generated by classes \( h_i, i \geq 0 \).
- \( h_j \) survives the Adams SS if \( \mathbb{R}^{2j} \) admits a division algebra structure.
Browder’s Reformulation

Theorem (Browder 1969)

There are no smooth Kervaire invariant one manifolds in dimensions not of the form $2^{j+1} - 2$. 
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1. There are no smooth Kervaire invariant one manifolds in dimensions not of the form $2^{j+1} - 2$.
2. There is such a manifold in dimension $2^{j+1} - 2$ iff $h_j^2$ survives the Adams spectral sequence.
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$$d_2(h_{j+1}) = h_0 h_j^2.$$
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**Theorem (Mahowald-Tangora)**

*The class $h_4^2$ survives the Adams SS.*
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**Theorem (Barratt-Jones-Mahowald)**

*The class $h_5^2$ survives the Adams SS.*
Previous Progress

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**Theorem (Barratt-Jones-Mahowald)**

The class $h_5^2$ survives the Adams SS.

**Theorem (H.-Hopkins-Ravenel)**

For $j \geq 7$, $h_j^2$ does not survive the Adams SS.
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1. Reduce to a simpler case which faithfully sees the Kervaire classes
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General Outline

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4. Show homotopy is periodic with period $2^8$
Reduction to Simpler Cases

Contains our classes $\text{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$

Adams SS
Reduction to Simpler Cases

Adams-Novikov SS

“More initial”
More complicated $\text{Ext}$

Adams SS

Contains our classes
$\text{Ext}_\mathcal{A}(F_2, F_2)$
Reduction to Simpler Cases

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Adams SS
Contains our classes
Extₐ(ℤ₂, ℤ₂)

HFP SS
Algebraically simple
H*(ℤ/8; R)

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Ext A(ℤ₂, ℤ₂)
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Passage from Adams to Adams-Novikov is well understood. Reduction from Adams-Novikov to homotopy fixed points is formal deformation theory. So good choice of $R$ gives us something that is
- easily computable
- strong enough to detect the classes.
Why Go Equivariant?

- Homotopy fixed point spectral sequence is still too complicated.
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- And there are spheres for every real representation.

Example

If $G = \mathbb{Z}/2$, then have $S^{\rho_2} = \mathbb{C}^+$ and $S^2$. 
Focus now on $G = \mathbb{Z}/8$. 

$RO(\mathbb{Z}/8)$ is rank 5 over $\mathbb{Z}$, generated by 1-dim reps: trivial rep $1$ and sign rep $\sigma$ and 2-dim reps: $L$, $L_2$, $L_3$. We care only about $\rho_{8} = 1 \oplus \sigma \oplus L \oplus L_2 \oplus L_3$. Plus the regular reps for subgroups.
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Important Representations

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1. Begin with $MU$ with $\mathbb{Z}/2$ given by complex conjugation.
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1
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\phantom{\text{MU}} \\
\text{MU} \boxtimes \text{MU} \boxtimes \text{MU} \boxtimes \text{MU} \\
\end{array}
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3. The “fixed points” for the $\mathbb{Z}/8$-action is geometric.
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   \end{array} \]

3. The “fixed points” for the $\mathbb{Z}/8$-action is geometric.
4. Inverting an equivariant class $\Delta$ makes the fixed points and homotopy fixed points agree.
Advantages of the Slice SS
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Basic Idea of Slices

Want to decompose $X$ into computable pieces.
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Acceptable Ones

- $S^{k_8}$, $S^{k_8-1}$

Unacceptable Ones

- $S^{k_8-2}$
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Acceptable Ones
1. $S^{k\rho_8}, S^{k\rho_8 - 1}$
2. $\mathbb{Z}/8_+ \wedge \mathbb{Z}/4 \ S^{k\rho_4}$

Unacceptable Ones
1. $S^{k\rho_8 - 2}$
2. $\mathbb{Z}/8_+ \wedge \mathbb{Z}/4 \ S^\sigma$
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4. $\mathbb{Z}/8_+ \wedge S^k$

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3. $\mathbb{Z}/8_+ \wedge \mathbb{Z}/2 \ S^{\sigma-1}$
4. $S^k$
Computing with Slices

Key Fact

For spectra like $MU$, slices can be computed from equivariant simple chain complexes.
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Cellular Chains for $S^{p_4-1}$
Gives the chain complex

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**Cellular Chains for $S^{ρ4−1}$**

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$$\mathbb{Z}^4 \to \mathbb{Z}^4 \to \mathbb{Z}^2 \to \mathbb{Z} = C.$$
Computing with Slices

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$$\mathbb{Z}^4 \rightarrow \mathbb{Z}^4 \rightarrow \mathbb{Z}^2 \rightarrow \mathbb{Z} = C_•.$$  

Maps determined by $H_*(C_•) = H_*(S^3)$.
For any non-trivial subgroup $H$ of $\mathbb{Z}/8$ and for any slice sphere $\mathbb{Z}/8_+ \wedge_H S^p$, 

$$H^{-2}(C_{\mathbb{Z}/8}^*) = 0$$
Theorem

For any non-trivial subgroup $H$ of $\mathbb{Z}/8$ and for any slice sphere $\mathbb{Z}/8_+ \wedge_H S^{p_H}$,

$$H_{-2}(C_*^{\mathbb{Z}/8}) = 0$$

The proof is an easy direct computation:
For any non-trivial subgroup $H$ of $\mathbb{Z}/8$ and for any slice sphere $\mathbb{Z}/8_+ \wedge_H S^{\rho_H}$,

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The proof is an easy direct computation:

1. If $k \geq 0$, then we are looking at something connected.
For any non-trivial subgroup \( H \) of \( \mathbb{Z}/8 \) and for any slice sphere \( \mathbb{Z}/8_+ \wedge \_ \mathbb{S}^\rho H \),
\[
H_{-2}(C_{\_}^{\mathbb{Z}/8}) = 0
\]

The proof is an easy direct computation:

1. If \( k \geq 0 \), then we are looking at something connected.
2. If \( k \leq 0 \), then we look at the associated cochain algebra.
Gaps

Theorem

For any non-trivial subgroup $H$ of $\mathbb{Z}/8$ and for any slice sphere $\mathbb{Z}/8_+ \wedge_H S^{\rho_H}$,

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The proof is an easy direct computation:

1. If $k \geq 0$, then we are looking at something connected.
2. If $k \leq 0$, then we look at the associated cochain algebra.
3. In the relevant degrees, the complex is $\mathbb{Z} \to \mathbb{Z}^2$ by $1 \mapsto (1,1)$. 

Theorem

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Gap Theorem

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Slices of \( MU \otimes MU \otimes MU \otimes MU \) are all of the form

\[ H\mathbb{Z} \otimes (\mathbb{Z}/8 \otimes_H S^{k\rho_H}). \]
Gap Theorem

**Theorem**

\[ \pi_{-2}(R) = 0. \]

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- Slices of \( MU \otimes MU \otimes MU \otimes MU \) are all of the form
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- Class we are inverting is carried by an \( S^{k\rho_8} \).
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\[ H\mathbb{Z} \otimes (\mathbb{Z}/8 \otimes_H S^{k\rho_H}). \]

- Class we are inverting is carried by an \( S^{k\rho_8} \).
- Inversion is a colimit and first steps show \( \pi_{-2} = 0. \)
Take Home Message
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Happy $A_5$ Birthday, Bob and Ron!