SYNMASKY: MORAVA STABIZER GROUP II

Note Title 3/19/2010

GOAL: Understand $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*(X))$ the E_2-term of the ANSS converging to $\pi_*(X_{(p)})$, e.g. $X = S^0$

$$\mathrm{Ext}_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n) \cong \mathrm{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*)$$

$\mathrm{Aut}^0_{\mathbb{F}_p}(F_n)$ = strict automorphism gp of height n FGL $/k$, where k is a field containing $\mathbb{F}_{p^n}$.

S_n = gp of units congruent to 1 mod S

Morava K-theory $K(n)_* = \mathbb{F}_p[v_n^{-1}, v_n]$

$|v_n| = 2(p^n-1)$ graded as

$$\Sigma(n) = K(n)_*[t_1, t_2, \dots]/(t_i^{p^n} - v_n^{p^i-1} t_i)$$

Consider the SES

$$\begin{tikzcd}
 & N^0 \arrow[d, phantom, "\parallel"] & M^0 \arrow[d, phantom, "\parallel"] & N^1 \arrow[d, phantom, "\parallel"] & \\
0 \arrow[r] & BP_* \arrow[r] & p^{-1}BP_* \arrow[r] \arrow[d, phantom, "\parallel"] & BP_*/p^\infty \arrow[r] \arrow[d, phantom, "\parallel"] & 0 \\
 & & BP_* \otimes \mathbb{Q} & \varinjlim BP_*/p^i \arrow[d, phantom, "\parallel"] & \\
 & & & BP_* \otimes \mathbb{Q}/\mathbb{Z}_{(p)} &
\end{tikzcd}$$

Another SES

$$0 \to N^1 \to v_1^{-1} N^1 \to N^2 \to 0$$

$$\vdots$$

$$0 \to N^n \to v_n^{-1} N^n \to N^{n+1} \to 0$$

So $N^n = BP_*/(p^\infty, v_1^\infty, \ldots, v_{n-1}^\infty)$

$M^n = v_n^{-1}$ "

These SESs can be spliced into a LES

$$0 \to BP_* \to M^0 \to M^1 \to \cdots,$$

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the chromatic resolution for BP_*.

NOTATION: $\mathrm{Ext}(M) := \mathrm{Ext}_{BP_* BP}(BP_*, M)$

for a BP_*BP-comodule.

$I_n = (p, v_1, \ldots, v_{n-1})$

There is a way to relate $\mathrm{Ext}(BP_*)$ to the grps $\mathrm{Ext}(M^n)$.

$$\mathrm{Ext}^{s,t} M^0 = \begin{cases} \mathbb{Q} & \text{for } (s,t) = (0,0) \\ 0 & \text{else} \end{cases}$$

Consider $M^1 = v_1^{-1} BP_* / (p^\infty)$

$$0 \to \underset{\underset{M_1^0}{\parallel}}{v_1^{-1} BP_* / p} \to M^1 \xrightarrow{p} M^1 \to 0$$

$$\mathrm{Ext}(M_1^0) = \mathrm{Ext}_{\Sigma(1)}(K(1)_*, K(1)_*)$$

$n=2 \quad \operatorname{Ext} M_2^0 = \operatorname{Ext}_{\Sigma(2)}\left(K(2)_*, K(2)_*\right)$

$$0 \to M_1^1 \longrightarrow M_0^2 \xrightarrow{p} M_0^2 \longrightarrow 0$$

$$M_0^2 = v_2^{-1} BP_* / (p^\infty, v_1^\infty)$$

$$0 \to M_2^0 \longrightarrow M_1^1 \xrightarrow{v_1} M_1^1 \longrightarrow 0$$

Cohomological properties of S_n:

- Explicitly calculated for $n=0,1,2$ and for $n=3$, $p \geq 5$.
- Finitely generated as an algebra.
- $\operatorname{Ext}^s_{\Sigma(n)} = 0$ for $s > n^2$ when $(p-1) \nmid n$.

- When $(p-1)|n$, S_n has an elt of order p and $\mathrm{Ext}^s \neq 0$ $\forall s \geq 0$.
- Ext is periodic: $\exists x \in H^{2i}(S_n)$ for some $i > 0$ such that $H^*(S_n)$ is a f g free module over $\mathbb{Z}/p[x]$
- if $p^i(p-1)|n$, S_n has an elt of order p^{i+1} and $\mathrm{Ext}^s \neq 0$ $\forall s \geq 0$

Relevant example is $(p,n) = (2,4)$ order 8.

Consider a finite spectrum X

$\underline{Q}$ Do we still get exactness

① If X is finite with no torsion in H_*, then $BP_*(X) = BP_* \otimes H_*(X)$ is a free BP_*-module that comes equipped with a BP_*BP-comodule structure. Tensoring with chrom resoln ① preserves exactness.

What if $H_* X$ has torsion? e.g.

$$0 \to S^0 \xrightarrow{p} S^0 \longrightarrow S^0 \cup e^1 \longrightarrow 0$$

$$S^0 \cup e^1 = V(0) \text{ or } M$$

Then $BP_*(X) = BP_*/p$.

Tensoring $BP_*(M)$ with (2) is <u>not</u> exact.

$$\to \overset{0}{\overset{\shortparallel}{\mathrm{Tor}^1(BP_*/p, BP_*)}} \longrightarrow \overset{0}{\overset{\shortparallel}{\mathrm{Tor}^1(BP_*/p, M^0)}} \to \mathrm{Tor}^1(BP_*/p, N^1) \to BP_*/p \longrightarrow 0$$

$$\mathrm{Tor}^1(BP_*/p, M^n) = \begin{cases} 0 & n=0 \\ v_1^{-1}BP_*/p & n=1 \\ v_2^{-1}BP_*/(p, v_1^\infty) & n=2 \end{cases}$$

and $BP_*/p \otimes M^n = 0$ for all n.