Wilne's account is very useful. 

$G = SL_2 \mathbb{Z}$ (or a subgroup) acts on upper half plane and we get an orbit space. 

Will generalize both $G$ and $H$. 

Rong Definition: 

$D$ - Hermitian symmetric domain 

$\Gamma$ - congruence subgroup of an algebraic group.
A Shimuravary is a directed system of such orbit spaces, \( \mathcal{D} \) as \( T \) ranges over congruence subgroups.

It will be the moduli space for abelian varieties with certain degeneracies on polarizations.

This level of generality appears to be adequate for reading Behrens-Jawson.

A Hermitian symmetric domain is a real manifold with a Hermitian form on tangent space and some other stuff.
\[ Y(1) = \text{PSL}_2(\mathbb{Z}) \backslash \mathbb{H} \]

is a coarse moduli stack for elliptic curves. \[ \mathbb{C}P^1 - \mathbb{R} \xrightarrow{\sim} \mathbb{C} \]

Classical

\[ Y(n) = \Gamma(n) \backslash \mathbb{H} \]

where \( \Gamma(n) = \{ (a, b) \in \mathbb{Z}^2 \mid \text{gcd}(a, b) = 1 \} \mod n \)

\[ s \rightarrow \mathcal{H} \]

As \( s \) ranges over cong. subgps we get polarized abelian varieties.
Classical modular varieties:

\[ N = \{ z = x + iy : y > 0 \} \]

\[ \Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\} \]

The action on \( \mathbb{H} \) is via Möbius transformations:

\[ \gamma(z) = \frac{az + b}{cz + d} \]

\[ Y(N) = \text{coarse moduli space for elliptic curves with a cyclic } N \text{-isogeny} \]

An \textbf{isogeny} \( E \rightarrow E' \) is a regular homomorphism of finite type. An \textbf{isomorphic } N \text{-isogeny} if \( \ker \phi = \mathbb{Z}/N \).
Shimura curves

A quaternion algebra over $k$ is a 4-dimensional central simple algebra over $k$, i.e., center is $k$ and it is not a sum of smaller algebras. $H \otimes_k R$ is either the usual quaternionic $H$ (definite case) or $M_2(R)$ (indefinite case).

An order in $H$ is a lattice closed under scalar multiplication (free abelian group of rank 4), the analog of the ring of integers in a number field.
A maximal order $\Lambda$ is an order not properly contained in any other. A definite quaternion algebra can have non-conjugate maximal orders.

Fix $H$ and a maximal order $\Lambda \subset H$ with $H$ indefinite. Choose an embedding $\iota : H \hookrightarrow M_2(\mathbb{R}) = H \otimes \mathbb{R}$.

$\Lambda^\times \hookrightarrow \text{GL}_2(\mathbb{R})$

$\Lambda^\times$ acts on $\mathcal{O} / \mathcal{R} = \mathcal{H} \cup \overline{\mathcal{H}}$

$\Gamma = \text{Stab}(\mathcal{H})$ = norm 1 elements of $\Lambda^\times$

$\mathcal{H}$ is Shimura curve. This is a coarse moduli problem for abelian surfaces with quaternionic multiplication by $\Lambda$. 
General moduli problem.

Fix $B$, a semi-simple $k$-algebra, $k$ a *-field, an involution on $B$.

$V$ a symplectic $(B,*)$-module, i.e. we have $\psi(b \cdot u, v) = \psi(u, b^* v)$.

Let $G$ be the symplectic automorphisms of $V$:

$$G = \{ g \in \text{Aut}_B(V) \mid \psi(gu, gw) = \mu(g) \psi(u, w) \}$$

with $\mu(g) \in k^\times$.

$X =$ set of complex structures on $V(R)$ that are positive or negative, meaning
Y is positive or negative definite in $G \backslash \Gamma$ is a Shimura vty of PEL type.

It solves the moduli problem:

- A an abelian vty
- a polarization (morphism $\lambda : A \to A^\vee$)
- a hom $B \to \text{End}_Q(A) \otimes Q$

restricting to an inclusion

$\Omega_B \subset \Lambda \subset \text{End}_Q(A)$, maximal order, with an additional condition