Note Title 10/30/2009

TAF Rogers on Shimura varieties

Milne's account is very useful.

$G = SL_2\mathbb{Z}$ (or a subgp) acts on upper half plane $\mathcal{H}$ and we get an orbit space

Will generalize both G and $\mathcal{H}$.

Rough definition:

D = Hermitian symmetric domain

Γ = congruence subgp of an algebraic gp.

A *Shimura vty* is a directed system of such orbit spaces $\Gamma \backslash D$ as Γ ranges over congruence subgps.

It will be the moduli space for abelian vties with certain isogenies or polarizations.

This level of generality appears to be adequate for reading Behrens–Lawson.

A *Hermitian symmetric domain* is a real manifold with a Hermitian form on tangent space + some other stuff.

$Y(1) = PSL_2(\mathbb{Z}) \backslash \mathcal{H}$ is a coarse moduli stack for elliptic curves $= \mathbb{CP}^1 - \{3 \text{ pts}\}$

<u>Classical</u>

$Y(1)$

$Y(N) = \Gamma(N) \backslash \mathcal{H}$

where $\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}$

$S = \Gamma \backslash D$

As Γ ranges over cong. subgps we get polarized abelian vties.

<u>Topological</u>

Moduli stack of ell. curves

Shimura Stacks

① Classical modular varieties:

$$\mathbb{H} = \{ z = x + iy : y > 0 \}$$

$$\Gamma(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \gamma \equiv I \mod N \right\} / \{\pm I\}$$

action on $\mathbb{H}$ is via Möbius transformations

$$\gamma(z) = \frac{az+b}{cz+d}$$

$Y(N)$ = coarse moduli space for elliptic curves with a cyclic N-isogeny

An isogeny $E \to E'$ is a regular hom of finite type.

It is a cyclic N-isogeny if $\ker \phi = \mathbb{Z}/N$.

(2) Shimura curves

k = real number field

A *quaternion algebra* H *over* k is a 4-dimensional central simple algebra over k, i.e. center is k and it is not a sum of smaller algebras. $H \otimes_k \mathbb{R}$ is either the usual quaternions $\mathbb{H}$ (definite case)

or $M_2(\mathbb{R})$ (indefinite case)

An *order* in H is a lattice closed under mult (free abelian gp of rank 4), the analog of the ring of integers in a # field

A maximal order Λ is an order not properly contained in any other. A definite quat. algebra can have nonconjugate maximal orders.

Fix H and a max. order $\Lambda \subset H$ with H indefinite.

Choose an embedding $j : H \hookrightarrow M_2(\mathbb{R}) = H \otimes \mathbb{R}$

$$\Lambda^\times \hookrightarrow GL_2(\mathbb{R})$$

$\Lambda^\times$ acts on $\mathbb{C} \setminus \mathbb{R} = \mathcal{H} \cup \overline{\mathcal{H}}$

$\Gamma = \mathrm{Stab}(\mathcal{H}) =$ norm 1 elements of $\Lambda^\times$

$\Gamma \backslash \mathcal{H}$ is Shimura curve. This is a coarse moduli problem for abelian surfaces with quaternionic mult by Λ.

General moduli problem:

Fix B, a semi-simple k-algebra; k a #-field

$*$ an involution on B

V a symplectic $(B, *)$ module: i.e.

we have $\psi(bu, v) = \psi(u, b^* v)$

Let G be the symplectic automorphisms of V

$$G = \{ g \in \mathrm{Aut}_B(V) \mid \psi(gu, gv) = \mu(g)\psi(u,v) \}$$

with $\mu(g) \in k^\times$

X = set of complex structures on $V(\mathbb{R})$ that are positive or negative, meaning

Ψ is positive or negative definite.

$\lim G \backslash X$ is a Shimura vty of PEL type.

It solves the moduli problem:

- A an abelian vty
- a polarizn (isogeny $A \longrightarrow A^\vee$)
- a hom $B \to \mathrm{End}(A) \otimes \mathbb{Q}$ restricting to an inclusion $\mathcal{O}_B \hookrightarrow \Lambda \subset \mathrm{End}(A)$ maximal order, with an additional condition