THE HOMOTOPY GROUPS OF tmf AND OF ITS LOCALIZATIONS

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In this small survey, we present a compilation of the homotopy groups of tmf and of its various localizations. This work should be thought of as an exercise in “collecting some of the diffuse knowledge” from my mathematical surroundings.

1. THE HOMOTOPY OF tmf

The spectrum tmf is connective, which means that the ring \( \pi_n(tmf) \) is zero for \( n < 0 \). Vaguely speaking, its homotopy ring \( \pi_*(tmf) \) is an amalgam of \( MF_* = \mathbb{Z}[c_4, c_6, \Delta]/(c_4^2 - (12)^3 \Delta) \), the ring of classical modular forms, and the ring \( \pi_*(S) \) of stable homotopy groups of spheres. Namely, there are ring homomorphisms

\[
\pi_*(S) \rightarrow \pi_*(tmf) \rightarrow MF_*
\]

that we now describe. Both maps are surprisingly close to being isomorphisms (even though \( \pi_*(S) \) and \( MF_* \) have nothing to do with each other).

The first map (1) is the Hurewicz homomorphism: being a ring spectrum, \( tmf \) admits a unit map from the sphere spectrum \( S \). This induces a map in homotopy \( \pi_*(S) \rightarrow \pi_*(tmf) \), which is an isomorphism on \( \pi_0 \). The only torsion in \( \pi_*(tmf) \) 2-torsion and 3-torsion, and so its is only at those primes that \( tmf \) resembles the sphere spectrum. The 3-primary part of the Hurewicz image is 72-periodic and is given by

\[
\text{im} \left( \pi_*(S) \rightarrow \pi_*(tmf) \right)_{(3)} = \mathbb{Z}(3) \oplus \alpha \mathbb{Z}/3\mathbb{Z} \oplus \bigoplus_{k \geq 0} \Delta^{3k} \{ \beta, \alpha \beta, \beta^2, \beta^3, \beta^4/\alpha, \beta^4 \} \mathbb{Z}/3\mathbb{Z},
\]

where \( \alpha \) has degree 3, and \( \beta = (\alpha, \alpha, \alpha) \) has degree 10. It doesn’t contains all the 3-torsion of \( \pi_*(tmf) \) as the classes in dimensions \( 27 + k \cdot 72 \) and \( 75 + k \cdot 72 \) for \( k \geq 0 \) are not hit by elements of \( \pi_*(S) \). The 2-torsion of \( \text{im}(\pi_*(S) \rightarrow \pi_*(tmf)) \) is much more complicated. It exhibits very rich patterns including two distinct periodicity phenomena. The first one is a periodicity by \( c_4 \in \pi_8(tmf) \) which corresponds to \( v_1^2 \), and the second one is a periodicity by \( \Delta^8 \in \pi_{192}(tmf) \) which corresponds to \( v_2^2 \).

The second map (1) can be described as the boundary homomorphism of the elliptic spectral sequence. Under that map, a class in \( \pi_n(tmf) \) maps to a modular form of weight \( n/2 \) (and maps to zero if \( n \) is odd). That map is an isomorphism after inverting the primes 2 and 3, which means that both its kernel and its cokernel are 2- and 3- torsion. The cokernel can be described explicitly

\[
\text{coker} \left( \pi_n(tmf) \rightarrow MF_* \right) = \begin{cases} \mathbb{Z}/24\mathbb{Z}, & \text{if } n = 24k \\ (\mathbb{Z}/2\mathbb{Z})^{(n-8)/4}, & \text{if } n \equiv 4 \pmod{8} \\ 0, & \text{otherwise.} \end{cases}
\]

The first cyclic group is generated by \( \Delta^k \), while the second group is generated by \( \Delta^b c_0 \) for integers \( a \) and \( b \) satisfying \( 24a + 8b + 12 = n \). The kernel agrees with the torsion in \( \pi_*(tmf) \), and it is much more complicated since it comes from the stable homotopy groups of spheres. Its 3-primary component is at most \( \mathbb{Z}/3\mathbb{Z} \) in any given degree. Its two primary component is a direct sum of \( (\mathbb{Z}/2\mathbb{Z})^\ell \) for some \( \ell \) (corresponding the \( v_1 \)-periodic elements) with a group isomorphic to \( \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/8\mathbb{Z}, \text{or } (\mathbb{Z}/2\mathbb{Z})^2 \) (corresponding the \( v_2 \)-periodic elements).
The picture on the left of this page represents the homotopy ring of $\text{tmf}$ at the prime 2. The vertical direction has no meaning. Bullets represent $\mathbb{Z}/2\mathbb{Z}$'s while squares represent $\mathbb{Z}/2^\infty\mathbb{Z}$. A chain of $n$ bullets connected by vertical lines represent a $\mathbb{Z}/2^n\mathbb{Z}$.

The bullets are named by the classes in $\pi_*(\text{S})$ of which they are the image ($\eta$, $\nu$, $\varepsilon$, $\kappa$, $\bar{q}$ are standard names), while the squares are named by their images in $\text{MF}_*$. The slanted lines represent multiplication by $\eta$, $\nu$, $\varepsilon$, $\kappa$, and $\bar{q}$.

The 20th part of the diagram is 192-periodic with polynomial generator $\Delta^8$. We have colored the image of the Hurewitz homomorphism [conjectured by Mark Mahowald] as follows: the $v_1$-periodic classes are in green, and the $v_2$-periodic classes are in pink, red, and blue, depending on their periodicity. The green classes are $v_1$-periodic in the sphere, and, except for $\nu$, they remain periodic in $\text{tmf}$ via the identification $c_4 = v_1^4$. The $v_2^1$-periodic classes are pink, the $v_2^1$-periodic red, and the $v_2^{32}$-periodic blue. They remain periodic in $\text{tmf}$ via the identification $\Delta^8 = v_2^2$.

The white numbers written in the squares indicate the size of $\text{coker}(\pi_*(\text{tmf}) \to \text{MF}_*)$. The $\mathbb{Z}/(2)$-algebra $\pi_*(\text{tmf})/(2)$ is finitely generated, with generators

| degree: | 1 3 8 8 12 14 20 24 25 27 32 32 | name: | $\eta$ $\nu$ $c_4$ $\varepsilon$ $(2\nu_6)$ $\kappa$ $\bar{q}$ $(8\Delta)$ $(2\nu\Delta)$ $q$ $(2c_6\Delta)$ $(2c_4\Delta)$ $(2c_8\Delta)$ $(2c_4\Delta)$ $(2c_6\Delta)$ $(2c_8\Delta)$ | $\{2c_6\Delta\}$ | $\{4\Delta^2\}$ | $\{\nu\Delta^2\}$ | $\{c_4\Delta^2\}$ | $\{2c_8\Delta\}$ | $\{8\Delta^3\}$ | $\{c_4\Delta^3\}$ | $\{2c_6\Delta^3\}$ |
|--------|-------------------------------|-------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
|        | 36 48 51 56 60 72 80 84       |       |                  |                  |                  |                  |                  |                  |                  |                  |
|        | 96 97 99 104 108 110 120     |       |                  |                  |                  |                  |                  |                  |                  |                  |
|        | 123 128 132 144 147 152 156 |       |                  |                  |                  |                  |                  |                  |                  |                  |
|        | 168 176 180 192             |       |                  |                  |                  |                  |                  |                  |                  |                  |
| $\{\nu\Delta\}$ | $\{4\Delta^2\}$ | $\{\nu\Delta^2\}$ | $\{c_4\Delta^2\}$ | $\{2c_6\Delta\}$ | $\{8\Delta^3\}$ | $\{c_4\Delta^3\}$ | $\{2c_6\Delta^3\}$ | $\{\Delta^6\}$ | $\{2c_6\Delta^6\}$ |
|        | 1, 2, 3, 8, 14, 15, 21, 22, 26, 98, 122, 123, 124, 125, 128, 130, 133, 136, 142. |
| $\{q\Delta\}$ | $\{2\nu\Delta\}$ | $\{\nu\Delta^2\}$ | $\{\nu\Delta^2\}$ | $\nu\Delta^2$ | $\nu\Delta^2$ | $\nu\Delta^2$ | $\nu\Delta^2$ | $\nu\Delta^2$ | $\nu\Delta^2$ |
|        | 1, 2, 3, 4, 8, 14, 15, 20, 21, 25, 26, 27, 28, 32, 34, 35, 40, 41, 45, 50, 60, 65, 75, 80, 85, 97, 98, 99, 100, 104, 105, 110, 111, 113, 117, 122, 123, 124, 125, 128, 130, 131, 137. |
|        | $\nu^2\Delta^2 \to \nu^2\Delta^2\}$ | $\nu^2\Delta^2 \to \nu^2\Delta^2\}$ | $\nu^2\Delta^2 \to \nu^2\Delta^2\}$ | $\nu^2\Delta^2 \to \nu^2\Delta^2\}$ | $\nu^2\Delta^2 \to \nu^2\Delta^2\}$ | $\nu^2\Delta^2 \to \nu^2\Delta^2\}$ | $\nu^2\Delta^2 \to \nu^2\Delta^2\}$ | $\nu^2\Delta^2 \to \nu^2\Delta^2\}$ | $\nu^2\Delta^2 \to \nu^2\Delta^2\}$ |

All the remaining multiplications are implied by the fact that $\pi_*(\text{tmf}) \to \text{MF}_*$ is a ring homomorphism.

To finish, we list the only two relations that are not implied by our chart: $\{\eta\Delta\} = \bar{q}$, $\{2\nu\Delta\} = \kappa\bar{q}$.
The picture of $\pi_* (tmf)_{(2)}$ from the previous page being rather small, we include here a larger version:

The homotopy groups of $tmf$ at the prime 3 exhibit similar phenomena than at the prime 2. The picture on the bottom of this page is an illustration of $\pi_* (tmf)_{(3)}$. The bullets represent $\mathbb{Z}/3\mathbb{Z}$'s and are named after the elements of $\pi_* (S)$ that hit them. The squares represent $\mathbb{Z}$'s and are named after their image in $MF_*$. The slanted lines represent multiplication by $\alpha$ and $\beta$. The top part
of the diagram is 72-periodic, with polynomial generator \( \Delta^3 \). We have drawn the image of the Hurewitz in color: in green is the unique \( v_1 \)-periodic class \( \alpha \), and in red are the \( v_2 \)-periodic classes. The latter remain periodic in \( \text{tmf} \) through the identification \( v_2^0 = \Delta^6 \) (or maybe \( v_2^{0/2} = \Delta^3 \) ?). Once again, the white numbers in the squares indicate the size of \( \text{coker}(\pi_* (\text{tmf}) \to M\text{F}_*)_{(3)} \). The algebra \( \pi_* (\text{tmf})_{(3)} \) is a finitely generated, with generators

\[
\begin{align*}
\text{degree:} & \quad 3 & 8 & 10 & 12 & 24 & 27 & 32 & 36 & 48 & 56 & 60 & 72 \\
\text{name:} & \quad \alpha & c_4 & \beta & c_6 & (3 \Delta) & \{ \alpha \Delta \} & \{ c_4 \Delta \} & \{ c_6 \Delta \} & \{ 3 \Delta^2 \} & \{ c_4 \Delta^2 \} & \{ c_6 \Delta^2 \} & \{ \Delta^3 \}
\end{align*}
\]

and many relations.

It is also worth while noting that the classes in dimensions 3, 13, 20, 30 (mod 72) support non-trivial \( \langle \alpha, \alpha, - \rangle \) Massey products.

When localized at a prime \( p \geq 5 \), the homotopy ring of \( \text{tmf} \) becomes isomorphic to \( M\text{F}_* \). Since \( \Delta \in M\text{F}_{12} \) is a \( \mathbb{Z}_p \)-linear combination of \( c_4^3 \) and \( c_6^2 \), this further simplifies to \( \pi_* (\text{tmf})_{(p)} = \mathbb{Z}_p [c_4, c_6] \).

2. LOCALIZATIONS OF \( \text{tmf} \)

The periodic version of \( \text{tmf} \) goes by the name \( \text{TMF} \). It’s homotopy groups are given by

\[ \pi_* (\text{TMF}) = \pi_* (\text{tmf}) \cdot \{ \Delta^{24} \}^{-1} \cdot \mathbb{Z} \oplus \mathbb{Q} \cdot \text{trivial} \cdot \text{Massey products} \]

The homotopy groups \( \pi_* (\text{TMF}) \) are finitely generated, except for \( n \equiv 1, 2 \) (mod 8), in which case they contain a summand isomorphic to \( (\mathbb{Z}/2\mathbb{Z})^\infty \).

Fix a prime \( p \), and let \( K(n) \) denote the \( n \)th Morava K-theory at that prime (\( p \) is omitted from the notation). We can then consider the \( K(n) \)-localization \( L_{K(n)} \text{tmf} \) of the spectrum \( \text{tmf} \). The spectrum \( L_{K(0)} \text{tmf} \) is simply the rationalisation of \( \text{tmf} \) (and doesn’t depend on \( p \)). Its homotopy ring is therefore given by

\[ \pi_* (L_{K(0)} \text{tmf}) = \pi_* (\text{tmf}) \oplus \mathbb{Z} \cdot \mathbb{Q} \]

The homotopy groups of \( L_{K(1)} \text{tmf} \) are easiest to describe at the primes 2 and 3. In those cases, they are given by

\[ \pi_* (L_{K(1)} \text{tmf}) = \left( \pi_* (\text{KO}) \cdot [j^{-1}] \right)_p \]

\[ = \pi_* (\text{KO})_p \cdot [j^{-1}] \]

(3)

Here, the notation \( R(x) \) refers to powers series \( \sum_{k=0}^{\infty} a_k x^k \) whose coefficients \( a_k \in R \) tend to zero \( p \)-adically as \( k \to \infty \). The variable is called \( j^{-1} \) because its inverse corresponds to the \( j \)-invariant of elliptic curves. The reason why (3) is simpler at \( p = 2 \) and \( p = 3 \) is that at those primes there is a unique supersingular elliptic curve, and that its \( j \)-invariant is equal to zero. For general prime \( p \geq 3 \), let \( \alpha_1, \ldots, \alpha_n \) denote the supersingular \( j \)-values. Each element \( \alpha_i \) is a priori only an element of \( \mathbb{F}_{p^n} \) (actually in \( \mathbb{F}_{p^e} \)), however, their union \( S := \{ \alpha_1, \ldots, \alpha_n \} \) is always a scheme defined over \( \mathbb{F}_p \). Let \( \hat{S} \) denote any scheme over \( \mathbb{Z} \) whose reduction mod \( p \) is \( S \). The homotopy groups of \( L_{K(1)} \text{tmf} \) are then given by

\[ \pi_* (L_{K(1)} \text{tmf}) = \left( \text{functions on } \mathbb{P}^1 \setminus \hat{S} \right)_p \cdot \left[ b_{\Delta^3} \right], \quad \text{if } p \geq 3, \]

where \( b \) is in degree 4.

The homotopy ring of \( L_{K(2)} \text{tmf} \) is the completion of \( \pi_* \text{TMF} \) at the ideal generated by \( p \) and by the Hasse invariant \( E_{p-1} \):

\[ \pi_* (L_{K(2)} \text{tmf}) = \pi_* (\text{TMF})_{(p, E_{p-1})}, \quad \text{if } p \text{ arbitrary.} \]

The latter is a polynomial in \( c_4 \) and \( c_6 \) whose zeroes correspond to the supersingular elliptic curves. Once again, given the fact that there is a unique supersingular elliptic curve at \( p = 2 \) and 3, the above formula simplifies to

\[ \pi_* (L_{K(2)} \text{tmf}) = \pi_* (\text{TMF})_{(p, c_4)} \quad \text{if } p = 2, 3, \]

\[ = \pi_* (\text{TMF})_{(p, c_4)} \quad \text{if } p = 2, 3. \]
For $n > 2$, the localization $L_{K(n)}tmf$ is trivial, and thus satisfies $\pi_*(L_{K(n)}tmf) = 0$.

3. The Adams spectral sequence

The Adams spectral sequence for $tmf$ is a spectral sequence that converges to $\pi_*(tmf)_p$ for any given prime $p$. Its $E_2$ page is given by $\text{Ext}_{A_p}^{*,*}(\mathbb{F}_p, \mathbb{F}_p)$, where $A_p^{tmf}$ is a finite dimensional
$\mathbb{F}_p$-algebra which is a $\text{tmf}$-analog of the Steenrod algebra.

$$A^\text{tmf}_p := \text{hom}_{\text{tmf-modules}}(H^\cdot_{\mathbb{F}_p}, H_{\mathbb{F}_p})$$

At the prime 2, the natural map $A^\text{tmf}_2 \to A \cong A_2$ to the Steenrod algebra is injective. Its image is the subalgebra $A(2) \subset A$ generated by $Sq^1$, $Sq^2$ and $Sq^3$. That algebra is of dimension 64 over $\mathbb{F}_2$, and defined by the relations

$$Sq^1Sq^1 = 0, \quad Sq^2Sq^2 = Sq^1Sq^2Sq^1,$$
$$Sq^1Sq^4 + Sq^4Sq^1 = Sq^2Sq^1Sq^2 = 0, \quad \text{and}$$
$$Sq^4Sq^4 + Sq^2Sq^2Sq^4 + Sq^3Sq^1Sq^3 = 0.$$  

By the change of rings theorem, the Adams spectral sequence for $\text{tmf}$ can then be identified with the classical Adams spectral sequence

$$E_2 = \text{Ext}_A(H^\cdot(\text{tmf}), \mathbb{F}_2) = \text{Ext}_A(A/A(2), \mathbb{F}_2) \Rightarrow \pi_*(\text{tmf}).$$

The bigraded ring $\text{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$ is generated by the classes:

\begin{center}
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<th>2</th>
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<td>$h_2$</td>
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<td>15</td>
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\end{tabular}
\end{center}

subject to the following complete set of relations:

$$h_0h_1 = 0, \quad h_1h_2 = 0, \quad h_0^2h_2 = h_1^2, \quad h_0h_2^2 = 0, \quad h_2^2 = 0, \quad h_0c_0 = 0, \quad h_1c_0 = 0, \quad h_2c_0 = 0, \quad c_0^2 = 0, \quad c_0d_0 = 0,$$
$$c_0e_0 = 0, \quad c_0g = 0, \quad c_0h = h_0^2g, \quad c_0\beta = 0, \quad h_0\beta = h_2^2w_1, \quad h_1d_0 = h_0^2\beta, \quad h_2d_0 = h_0c_0, \quad d_0^2 = w_1g, \quad d_0g = c_0^2,$$
$$h_0\alpha = h_2^2\beta w_1, \quad \alpha^2d_0 = \beta^2w_1, \quad \beta d_0 = \alpha e_0, \quad h_1e_0 = h_2^2\alpha, \quad h_2e_0 = h_0g, \quad \beta e_0 = \alpha g, \quad h_1g = h_2^2\beta, \quad h_2g = 0, \quad c_0g = \alpha \gamma, \quad g^2 = \beta \gamma, \quad h_1\alpha = 0, \quad h_1\beta = 0, \quad h_2\alpha = h_0\beta,$$
$$h_0\beta^2 = 0, \quad h_2\beta^2 = 0, \quad \alpha^4 = h_0^3w_2 + g^2w_1, \quad h_0\gamma = 0, \quad h_2\gamma = h_2^2\gamma, \quad h_2\gamma = 0, \quad c_0\gamma = h_1\delta, \quad d_0\gamma = \alpha^2\beta,$$
$$h_0^3\gamma = \alpha \beta^2, \quad g^2 = \beta^3, \quad \gamma^2 = h_2^2w_2 + \beta^2g, \quad h_0\delta = h_2^2og, \quad h_2^2\delta = h_0d_0g, \quad h_2\delta = 0, \quad c_0\delta = 0, \quad d_0\delta = 0, \quad \alpha \delta = 0, \quad g^2 = 0, \quad \alpha \delta = 0, \quad \beta \delta = 0, \quad \gamma \delta = h_1c_0w_2, \quad \delta^2 = 0.$$  

The charts on the previous page only go up until the periodicity. Therefore, we have also included in the picture of the $E_\infty$ page the most important differential in dimensions $\geq 96$.

At the prime 3, the map from $A^\text{tmf}_3$ to the Steenrod algebra is no longer injective. Indeed, the algebra $A^\text{tmf}_3$ is 24 dimensional, while its image in the Steenrod algebra is the 12 dimensional subalgebra generated by $\beta$ and $P^1$. Naming its generators by their image in $A_3$, the following relations define $A^\text{tmf}_3$:

$$\beta^2 = 0, \quad (P^1)^3 = 0,$$

$$\beta P^1 \beta P^1 + P^1 \beta P^1 \beta = \beta (P^1)^2 \beta.$$
Note that the relation $\beta(\mathcal{P}^1)^2 + \mathcal{P}^1 \beta \mathcal{P}^1 + (\mathcal{P}^1)^2 \beta = 0$ holds in $A_3$, but not in $A_3^{tmf}$.

The Adams spectral sequence $\text{Ext}_{A_3^{tmf}}(\mathbb{F}_3, \mathbb{F}_3) \Rightarrow \pi_*(tmf)_3$.

At primes $p > 3$, the algebra $A_p^{tmf}$ is an exterior algebra on generators in degrees 1, 9, and 13. The ring $\text{Ext}_{A_p^{tmf}}(\mathbb{F}_p, \mathbb{F}_p)$ a polynomial algebra on classes in bidegrees (0, 1), (8, 1), and (12, 1) and the Adams spectral sequence for $tmf$ collapses.

The Adams spectral sequence for $tmf$ at a prime $p > 3$.

It is interesting to note that regardless of the prime, the algebra $A_p^{tmf}$ has its top dimensional class in degree 23. Below, we picture the algebras $A_p^{tmf}$ for the primes 2 and 3.
4. ACKNOWLEDGEMENTS

I am seriously indebted to many people for the completion of this survey on the homotopy groups of tmf. First of all, I would like to thank Tilman Bauer for spending an enormous amount of time with me sorting out the multiplicative structure of $\pi_\ast(tmf)$. Based on a computer program of Christian Nassau, Tilman is also the one who created the first draft of the Ext$_{A(2)}$ charts. I should thank Mark Behrens for sending me his calculations of the Adams spectral sequence, and for answering many questions about the multiplicative structure of $\pi_\ast(tmf)$. The defining relations of Ext$_{A(2)}$ were taken from a calculation of Peter May, who was made available to me by Mark Behrens. Next, I would like to thank Mark Mahowald for some very informative email exchanges. I am especially grateful to him for explaining me how to detect $v_{32}^2$-periodic families of elements in the image of Hurewitz homomorphism, and for agreeing to make the following conjecture:

**Conjecture** (Mark Mahowald). The image of the Hurewitz homomorphism $\pi_\ast(S) \to \pi_\ast(tmf)$ is given by the classes drawn in color in the pictures on page 3.

I thank Niko Nauman for useful conversations. I am indebted to Mike Hopkins for putting me in contact with Mark Mahowald. Finally, I should also thank Mike Hill, who was the first to figure out the structure of $A^{tmf}_4$, for the numerous conversations that we had while we were graduate students at MIT.