A geometric solution of the Kervaire Invariant One problem

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Let 
\[ f : M^{n-1} \hookrightarrow \mathbb{R}^n, \]

\[ n = 4k + 2, \ n \geq 2 \] be a smooth generic immersion of a closed manifold of codimension 1. Let 
\[ g : N^{n-2} \hookrightarrow \mathbb{R}^n \]

be the immersion of the double points intersection of \( g \).

The **Kervaire invariant** of \( f \) is defined by

\[ \Theta_{sf}(f) = \langle \eta_{N^{n-2}}^{n-2}; [N^{n-2}] \rangle, \]

where \( \eta_N = w_2(N^{n-2}) \) is the second normal Stiefel-Whitney class of \( N^{n-2} \).
In particular, if $n = 2$, $\Theta_{sf}(f)$ is the parity of the number of self-intersection points of the curve $f$ on the plane $\mathbb{R}^2$.

Let us denote by $Imm^{sf}(n - 1, 1)$ the cobordism group of immersions in the codimension 1 of (non-oriented) closed $(n - 1)$–manifold ("sf" stands for skew-framed).
Theorem

The Kervaire invariant is a well-defined homomorphism:

\[ \Theta_{sf} : Imm^{sf}(4k + 1, 1) \longrightarrow \mathbb{Z}/2. \]

1. This homomorphism is trivial if \( 4k + 2 \neq 2^l - 2, l \geq 2 \).
2. For \( k = 0, 1, 3, 7, 15 \) the homomorphism \( \Theta_{sf} \) is an epimorphism.

Main Theorem

There exists an integer $l_0$, such that for an arbitrary $l \geq l_0$, the Kervaire invariant

$$\Theta_{s_f} : Imm^{s_f}(2^l - 3, 1) \rightarrow \mathbb{Z}/2$$

is the trivial homomorphism.

A commutative diagram that we use to define dihedral structure of self-intersection manifold of skew-framed immersions:

\[\begin{array}{ccc}
\text{Imm}^{sf}(n-1, 1) & \xrightarrow{J_{sf}^k} & \text{Imm}^{sf}(n-k, k) \\
\downarrow \delta_{\mathbb{Z}/2[2]} & & \downarrow \delta_{\mathbb{Z}/2[2]}^k \\
\text{Imm}^{\mathbb{Z}/2[2]}(n-2, 2) & \xrightarrow{J_{\mathbb{Z}/2[2]}^k} & \text{Imm}^{\mathbb{Z}/2[2]}(n-2k, 2k).
\end{array}\]
A commutative diagram that we use to define the Kervaire invariant in the codimension $k$:

$$
\begin{array}{ccc}
\text{Imm}^s f(n - k, k) & \stackrel{\Theta^k f}{\longrightarrow} & \mathbb{Z}/2 \\
\downarrow \delta^k_{\mathbb{Z}/2[2]} & \quad & \\
\text{Imm}^n_{\mathbb{Z}/2[2]}(n - 2k, 2k) & \stackrel{\Theta^k_{\mathbb{Z}/2[2]}}{\longrightarrow} & \mathbb{Z}/2
\end{array}
$$

skew-framed immersions

dihedral immersions
Structure groups of immersions

Let us consider the following collection of \((d - 1)\) sets

\[ \Upsilon_d, \Upsilon_{d-1}, \ldots, \Upsilon_2, \]

where each set consists of proper coordinate subspaces of \(\mathbb{R}^{2d-1}\).

The set of the subspaces

\[ \Upsilon_i, \quad 2 \leq i \leq d, \]

(we will use only the case \(d = 6\)) consists of \(2^{i-1}\) coordinate subspaces generated by the basis vectors:

\[ ((e_1, \ldots, e_{2d-i}), \ldots, (e_{2d-1-2d-i+1}, \ldots, e_{2d-1})). \]
Let us denote by $\mathbb{Z}/2^d$ the subgroup

$$\mathbb{Z}/2 \wr \Sigma_{2^{d-1}} \subset O(2^{d-1})$$

under the following condition:

- the transformation

$$\mathbb{Z}/2^d \times \mathbb{R}^{2^{d-1}} \rightarrow \mathbb{R}^{2^{d-1}}$$

admits the invariant collection of sets

$$\Upsilon_d, \Upsilon_{d-1}, \ldots, \Upsilon_2.$$

In particular, in the case $d = 2$ we get that $\Upsilon_2$ contains only one collection of subspaces and this collection is $((\textbf{e}_1), (\textbf{e}_2))$. Therefore $\mathbb{Z}/2^2$ is a dihedral group.
\[ \text{Imm}^{\mathbb{Z}/2^2} (n - 2, 2) \rightarrow \text{Imm}^{\mathbb{Z}/2^2} (n - 2k, 2k) \]
\[ \downarrow \delta_{\mathbb{Z}/2^3} \quad \downarrow \delta_{\mathbb{Z}/2^3} \]
\[ \text{Imm}^{\mathbb{Z}/2^3} (n - 4, 4) \xrightarrow{J_{\mathbb{Z}/2^3}^k} \text{Imm}^{\mathbb{Z}/2^3} (n - 4k, 4k) \]
\[ \downarrow \delta_{\mathbb{Z}/2^4} \quad \downarrow \delta_{\mathbb{Z}/2^4} \]
\[ \text{Imm}^{\mathbb{Z}/2^4} (n - 8, 8) \xrightarrow{J_{\mathbb{Z}/2^4}^k} \text{Imm}^{\mathbb{Z}/2^4} (n - 8k, 8k) \]
\[ \downarrow \delta_{\mathbb{Z}/2^5} \quad \downarrow \delta_{\mathbb{Z}/2^5} \]
\[ \text{Imm}^{\mathbb{Z}/2^5} (n - 16, 16) \xrightarrow{J_{\mathbb{Z}/2^5}^k} \text{Imm}^{\mathbb{Z}/2^5} (n - 16k, 16k) \]
\[ \downarrow \delta_{\mathbb{Z}/2^6} \quad \downarrow \delta_{\mathbb{Z}/2^6} \]
\[ \text{Imm}^{\mathbb{Z}/2^6} (n - 32, 32) \xrightarrow{J_{\mathbb{Z}/2^6}^k} \text{Imm}^{\mathbb{Z}/2^6} (n - 32k, 32k). \]
\[
\begin{align*}
\text{Imm}^{Z/2^2}(n - 2k, 2k) & \rightarrow Z/2 \\
\downarrow \delta_{Z/2^2[k]}^k & \parallel \\
\text{Imm}^{Z/2^3}(n - 4k, 4k) & \rightarrow Z/2 \\
\downarrow \delta_{Z/2^3[k]}^k & \parallel \\
\text{Imm}^{Z/2^4}(n - 8k, 8k) & \rightarrow Z/2 \\
\downarrow \delta_{Z/2^4[k]}^k & \parallel \\
\text{Imm}^{Z/2^5}(n - 16k, 16k) & \rightarrow Z/2 \\
\downarrow \delta_{Z/2^5[k]}^k & \parallel \\
\text{Imm}^{Z/2^6}(n - 32k, 32k) & \rightarrow Z/2.
\end{align*}
\]
\[
\begin{array}{ccc}
I_d \oplus \dot{I}_d & \subset & \mathbb{Z}/2^{[2]} & \text{Abelian structure} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
I_a \oplus \dot{I}_d & \subset & \mathbb{Z}/2^{[3]} & \text{cyclic–Abelian structure} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
I_a \times \dot{I}_a & \subset & \mathbb{Z}/2^{[4]} & \text{bicyclic structure} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
Q \times \dot{I}_a & \subset & \mathbb{Z}/2^{[5]} & \text{quaternionic–cyclic structure} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
Q \times \dot{Q} & \subset & \mathbb{Z}/2^{[6]} & \text{biquaternionic structure}zes.
\end{array}
\]
Dihedral group

The dihedral group (of the order 8) $\mathbb{Z}/2[2] \subset O(2)$:

$$\{a, b \mid a^4 = b^2 = e, [a, b] = e^2\}.$$ 

Let $\{f_1, f_2\}$ be the standard base of the plane $\mathbb{R}^2$. The element $a$ is represented by the rotation through the angle $\frac{\pi}{2}$:

$$f_1 \mapsto f_2; \quad f_2 \mapsto -f_1.$$

The element $b$ is represented by the permutation of the base vectors

$$f_1 \mapsto f_2; \quad f_2 \mapsto f_1.$$
Elementary 2-group

The elementary subgroup $I_d \times \hat{I}_d \subset \mathbb{Z}/2[2]$ of the rank 2:

$$\{a^2, b \mid a^4 = b^2 = e, [a^2, b] = e\}.$$ 

This group preserves the vectors $f_1 + f_2$, $f_1 - f_2$.

Let $\tau_{[2]} \in H^2(\mathbb{Z}/2[2]; \mathbb{Z}/2)$ be the universal class, $i^*_{d \times d}(\tau_{[2]}) \in H^2(I_d \times \hat{I}_d; \mathbb{Z}/2)$ is the pull-back of $\tau_{[2]}$ under the inclusion $i_{d \times d} : I_d \times \hat{I}_d \subset \mathbb{Z}/2[2]$.

$$i^*_{d \times d}(\tau_{[2]}) = \kappa_d \kappa_{\hat{d}},$$

$\kappa_d \in H^1(I_d \times \hat{I}_d; \mathbb{Z}/2)$. $p_d : I_d \times \hat{I}_d \to I_d$, $\kappa_d = p_d^*(t_d)$, $e \neq t_d \in I_d \simeq \mathbb{Z}/2$, and $\kappa_{\hat{d}} \in H^1(I_d \times \hat{I}_d; \mathbb{Z}/2)$ is defined analogously.
\[ x = [(f, \Xi, \kappa_M)] \in Imm^s f(n - k, k), \]

\[-f : M^{n-k} \to \mathbb{R}^n, \]

\[-\kappa_M \text{ is a line bundle over } M^{n-k}, \]

\[-\Xi \text{ is a skew-framing of the normal bundle of } f, \text{ i.e. an isomorphism } \Xi : \nu_f = k\kappa_M. \]

\[ y = \delta^k_{\mathbb{Z}/2^2}(x) = (g, \Psi, \eta_N) \in Imm^{\mathbb{Z}[2]}(n - 2k, 2k), \]

\[-g : N^{n-2k} \to \mathbb{R}^n, \]

\[-\eta_N \text{ is a } \mathbb{Z}/2^2 \text{-bundle over } N^{n-2k}, \]

\[-\Psi \text{ is a dihedral framing of the normal bundle of } f, \text{ i.e. an isomorphism } \Psi : \nu_g = k\eta_N. \]
Definition of Abelian structure

A skew-framed immersion

$$(f, \Xi, \kappa_M)$$

admits an Abelian structure if there exists a map

$$\eta_{d \times d, N} : N^{n-2k} \to K(I_d \times I_d, 1)$$ (Eilenberg-Mac Lain space),

satisfying the following condition:

$$\langle \eta_{N}^{n-2k} ; [N^{n-2k}] \rangle = \Theta_{Z/2^2}^{k} (y) = \langle \eta_{N}^{15k} \eta_{d \times d}^{n-32k} ; [N] \rangle.$$

$$\eta_{d \times d} = \eta_{d \times d, N}^*(i_{d \times d}^*(\tau_{2^2})) \in H^2(N^{n-2k}; \mathbb{Z}/2), [N]$$ is the fundamental class of $N^{n-2k}$, $\eta_N \in H^2(N^{n-2k}; \mathbb{Z}/2)$ is the characteristic class of $\mathbb{Z}/2^{2^2}$–framing.
Definition of a Desuspension

A skew-framed cobordism class

\[ x = [(f, \Xi, \kappa_M)] \in Imm^{sf}(n - k, k) \]

admits a desuspension of order \( q \), if this class is represented by a triple, such that

\[ \kappa_M = I \circ \kappa(q), \]

\[ \kappa(q) : M^{n-k} \to \mathbb{R}P^{n-k-q-1}, I : \mathbb{R}P^{n-k-q-1} \subset \mathbb{R}P^\infty. \]
Desuspension theorem

For an arbitrary $q$ there exists an integer $l_0 = l_0(q)$, such that an arbitrary element $x \in Imm^{sf}(2^l - 3, 1), l \geq l_0,$ admits a desuspension of order $q$. 
Abelian structure immersion theorem

Let \( q \) be an arbitrary integer divisible by 16, and let 
\( n = 2^l - 2 \) with \( l \) is sufficiently large. Put 
\[
k = k(q) = \frac{n + 2}{32} - \frac{q}{16}.
\]

Let us assume that \( x \in Imm^s_f(n - k, k) \) admits a 
desuspension of order \( q \). Then the class \( x \) is represented 
by a triple \((f, \Xi, \kappa_M)\), such that this skew-framed 
immersion admits an Abelian structure.
Cyclic group

The cyclic index 2 subgroup of the order 4:

\[ I_a = \{a \mid a^4 = e\} \subset \mathbb{Z}/2^{[2]}. \]

Bicyclic group

The bicyclic index $2^{11}$ subgroup of the order 16:

\[ I_a \times \hat{I}_a \subset \mathbb{Z}/2^{[2]} \times \mathbb{Z}/2^{[2]} \subset \mathbb{Z}/2^{[4]}. \]
The universal cohomology class of the bicyclic group

There exists $\tau[4] \in H^8(\mathbb{Z}/2^4; \mathbb{Z}/2)$ (the universal class), $i^*_{a\times\dot{a}}(\tau[4]) \in H^8(I_a \times \dot{I}_a; \mathbb{Z}/2)$ is the pull-back of $\tau[4]$ under the inclusion $i_{a\times\dot{a}} : I_a \times \dot{I}_a \subset \mathbb{Z}/2^4$.

$$i^*_{a\times\dot{a}}(\tau[4]) = \eta^2_a \eta^2_{\dot{a}},$$

$\eta_a, \eta_{\dot{a}} \in H^2(I_a \times \dot{I}_a; \mathbb{Z}/2)$. Here $\eta_a, \eta_{\dot{a}}$ are defined similar to $\kappa_a, \kappa_{\dot{a}}$. 

\[x \in \text{Imm}^{sf}(n-k,k),\]
\[y = \delta_{\mathbb{Z}/2[2]}(x) \in \text{Imm}^{\mathbb{Z}/2[2]}(n-2k,2k),\]
\[z = \delta_{\mathbb{Z}/2[3]} \circ \delta_{\mathbb{Z}/2[2]} \in \text{Imm}^{\mathbb{Z}/2[3]}(n-4k,4k),\]
\[u = \delta_{\mathbb{Z}/4[4]} \circ \delta_{\mathbb{Z}/2[3]}(x)\]
\[u = [(h, \Lambda, \zeta_L)] \in \text{Imm}^{\mathbb{Z}/2[4]}(n-8k,8k),\]

- \(h : L^{n-8k} \rightarrow \mathbb{R}^n\),

- \(\zeta_L\) is a \(\mathbb{Z}/2[4]\)-bundle over \(L^{n-8k}\),

- \(\Lambda\) is a an 8-dimensional \(\mathbb{Z}/4[4]\)-framing of the normal bundle of \(h\), i.e. an isomorphism \(\Lambda : \nu_h \simeq k\zeta_L\).
Definition of bicyclic structure

A $\mathbb{Z}/2^3$–immersion

$$[(g', \Psi', \eta_{N'})] = \delta_{\mathbb{Z}/2^3} \circ \delta_{\mathbb{Z}/2^3}(x) \in Imm_{\mathbb{Z}/2^3}(n - 4k, 4k)$$

admits a bicyclic structure if there exists a map

$$\zeta_{a \times \hat{a}, L} : L^{n-8k} \to K(I_a \times \hat{I}_a, 1) \text{ (Eilenberg-Mac Lain space)},$$

satisfying the following equation:

$$\Theta^{k}_{\mathbb{Z}/2^3}(z) = \langle \pi_{d \times d, a \times \hat{a}, L}^*(\zeta_L)^{3k} \tilde{\zeta}_{d \times d, L}^{\frac{n-32k}{2}} \tilde{L}_{d \times d} \rangle.$$  

Here the cohomology class $\tilde{\zeta}_{d \times d, L}$ is defined by means of $\zeta_{a \times \hat{a}, L}$. 
In the previous formula:

- \( L^{n-8k} \) is the double-point \( \mathbb{Z}/2^4 \)-manifold of \( g' \)
- \( [\bar{L}_{d \times d}] \) is the fundamental class of the corresponding 4-sheeted cover

\[
\pi_{d \times d, a \times \hat{a}} : \bar{L}_{d \times d}^{n-8k} \to L^{n-8k},
\]

induced from the 4-sheeted cover of Eilenberg-Maclain spaces \( K(I_d \times \hat{I}_d, 1) \to K(I_a \times \hat{I}_a, 1) \) by the map

\[ \zeta_{a \times a, L} : L^{n-8k} \to K(I_a \times \hat{I}_a, 1) \]

- \( \bar{\zeta}_{d \times d, L} \in H^2(\bar{L}_{d \times d}^{n-8k}; \mathbb{Z}/2) \) is the universal cohomological \( I_d \times \hat{I}_d \)-class, constructed by means of the map \( \zeta_{Q \times \hat{Q}, L} \)
– $\zeta_L \in H^8(L^{n-8k};\mathbb{Z}/2)$ is the top characteristic class of the $\mathbb{Z}/2^{[4]}$–framing and it is the pull-back of the universal class $\tau_{[4]} \in H^8(\mathbb{Z}/2^{[4]};\mathbb{Z}/2)$ under the classifying map $L^{n-8k} \to K(\mathbb{Z}/2^{[4]},1)$ of the corresponded $\mathbb{Z}/2^{[4]}$–bundle previously denoted by $\zeta_L$.

– $\pi^*_{d\times d,a\times \hat{a}}(\zeta_L) \in H^8(\bar{L}_{d\times \hat{d}}^{n-8k};\mathbb{Z}/2)$ is the pull-back of the class $\zeta_L$ under the 4-sheeted cover

$$\pi^*_{d\times d,a\times \hat{a}} : \bar{L}_{d\times \hat{d}}^{n-8k} \to L^{n-8k}.$$
Bicyclic structure immersion Theorem

Let us assume that $x \in Imm^{sf}(n-k,k)$, $k = \frac{n-2^s+2}{32}$, $s \geq 6$, admits a desuspension of the order $q = \frac{2^s-2}{2}$. Then the class

$$z = \delta^k_{\mathbb{Z}[3]} \circ \delta^k_{[2]}(x) \in Imm^{\mathbb{Z}/2^{[4]}}(n-4k,4k)$$

is represented by a triple $(h, \Lambda, \zeta_L)$, such that this skew-framed immersion admits an Abelian structure.
Quatertionic group

The quaternionic group of the order 8:

\[ Q = \{ i, j, k \mid ij = k = -ji, jk = i = -kj, ki = j = -ik, i^2 = j^2 = k^2 = -1 \}. \]

This is an index 16 subgroup \( Q \subset \mathbb{Z}/2[3] \). The standard representation \( \chi_+ : Q \to \mathbb{Z}/2[3] \) transforms the quaternions \( i, j, k \) into the following matrices:
\[ i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \]
\[ j = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \]
\[ k = \begin{pmatrix}
  0 & 0 & 0 & -1 \\
  0 & 0 & -1 & 0 \\
  0 & 1 & 0 & 0 \\
  1 & 0 & 0 & 0 \\
\end{pmatrix}. \]
Biquaternionic group

The biquaternionic index $2^{57}$–subgroup of the order 64:

$$Q \times Q \subset \mathbb{Z}/2^{[3]} \times \mathbb{Z}/2^{[3]} \subset \mathbb{Z}/2^{[6]}.$$
The universal cohomology class of the biquaternionic group

There exists $\tau_{[6]} \in H^{32}(\mathbb{Z}/2^{[6]}; \mathbb{Z}/2)$ (the universal class), $i_*^* Q \times \dot{Q} (\tau_{[6]}) \in H^{32}(Q \times \dot{Q}; \mathbb{Z}/2)$ is the pull-back of $\tau_{[6]}$ under the inclusion $i_{Q \times \dot{Q}} : Q \times \dot{Q} \subset \mathbb{Z}/2^{[6]}$.

$$i_*^* Q \times \dot{Q} (\tau_{[6]}) = \zeta_Q \zeta_{\dot{Q}},$$

$\zeta_Q, \zeta_{\dot{Q}} \in H^4(Q \times \dot{Q}; \mathbb{Z}/2)$. Here $\zeta_Q, \zeta_{\dot{Q}}$ are defined similar to $\eta_a, \eta_{\dot{a}}$. 

\[ x \in \text{Imm}^s f(n-k,k), \]
\[ y = \delta_{Z/2[2]}(x) \in \text{Imm}^{Z/2[2]}(n-2k,2k), \]
\[ z = \delta_{Z/2[2]} \circ \delta_{Z/2[3]} \in \text{Imm}^{Z/2[3]}(n-4k,4k), \]
\[ u = \delta_{Z[4]} \circ \delta_{Z[3]} \circ \delta_{Z[2]}(x) \in \text{Imm}^{Z/2[4]}(n-4k,4k), \]
\[ v = \delta_{Z[4]} \circ \delta_{Z[3]} \circ \delta_{Z[2]} \circ \delta_{Z[3]}(x) \in \text{Imm}^{Z/2[4]}(n-8k,8k), \]
\[ w = \delta_{Z[6]} \circ \delta_{Z[5]} \circ \delta_{Z[4]} \circ \delta_{Z[3]} \circ \delta_{Z[2]}(x) \in \text{Imm}^{Z/2[5]}(n-16k,16k). \]
Definition of biquaternionic structure

A $\mathbb{Z}/2^{[5]}$–immersion $[(h', \Lambda', \zeta')] =$

$\delta_{\mathbb{Z}/2^{[5]}} \circ \delta_{\mathbb{Z}/2^{[4]}} \circ \delta_{\mathbb{Z}/2^{[3]}} \circ \delta_{\mathbb{Z}/2^{[2]}}(x) \in \text{Imm}^{\mathbb{Z}/2^{[5]}}(n - 16k, 16k)$

admits a biquaternionic structure if there exists a map

$\omega_{Q \times Q,K} : K^{n-32k} \to K(Q \times \dot{Q}, 1)$ (Eilenberg-Mac Lain space),

satisfying the following equation:

$\Theta^k_{\mathbb{Z}/2^{[5]}}(w) = \langle \bar{\omega}_{d \times d,K}^{n-32k}, [K_{d \times d}] \rangle$.

Here the cohomology class $\bar{\omega}_{d \times d,K}$ is defined by means of $\omega_{Q \times Q,K}$. 
In the previous formula:

- $K^{n-32k}$ is the double-point $\mathbb{Z}/2^6$–manifold of $h'$
- $[\tilde{K}_{d\times d}]$ is the fundamental class of the 16-sheeted cover

$$\pi_{d\times d, Q\times Q}: \tilde{K}^{n-32k}_{d\times d} \to K^{n-32k},$$

induced from the 16-sheeted cover of Eilenberg-Mac Lain spaces $K(I_d \times \hat{I}_d, 1) \to K(Q \times \hat{Q}, 1)$ by the map

$\omega_{Q\times Q, K}: K^{n-32k} \to K(Q \times \hat{Q}, 1)$

- $\tilde{\omega}_{d\times d, K} \in H^2(\tilde{K}^{n-32k}_{d\times d}, \mathbb{Z}/2)$ is the universal cohomology $I_d \times \hat{I}_d$-class, constructed by means of the map $\omega_{Q\times Q, K}$. 
Biquaternionic structure immersion theorem

Let \( k = \frac{n-2s+2}{32}, \ s \geq 6, \ (q = \frac{2s-2}{2}), \ q \) be an integer divisible by 16, and let \( n = 2^l - 2 \) with \( l \) sufficiently large.

Put

\[
k = k(q) = \frac{n + 2}{32} - \frac{q}{16}.
\]

Let us assume that \( x \in Imm^s_f(n-k,k) \) admits a desuspension of the order \( q = \frac{2s-2}{2} \). Then the class \( w = \delta_{\mathbb{Z}^5}^k \circ \delta_{\mathbb{Z}^4}^k \circ \delta_{\mathbb{Z}^3}^k \circ \delta_{\mathbb{Z}^2}^k(x) \in Imm^{\mathbb{Z}/2^5}(n-16k,16k) \) is represented by a triple \( (h', \Lambda', \zeta_{L'}) \) such that this triple admits a biquaternionic structure.
Biquaternionic Kervaire Invariant Theorem

Assume that \( w \in \text{Imm}_{\mathbb{Z}/2^{[\xi]}}(n - 16k, 16k), n = 2^l - 2, \)
\( k \equiv 0 \pmod{64}, k > 0, n - 32k > 0 \) admits a
biquaternionic structure. Then \( \Theta_{\mathbb{Z}/2^{[\xi]}}(w) = 0. \)
As a corollary we get

Main Theorem

There exists an integer \( l_0, \) such that for an arbitrary
\( l \geq l_0, \) the Kervaire invariant
\[ \Theta_{sf} : \text{Imm}^{sf}(2^l - 3, 1) \longrightarrow \mathbb{Z}/2 \]
is the trivial homomorphism.
Proof of Biquaternionic Theorem

Let \( w \in \text{Imm}^{Z/2^{[5]}}(n-16k, 16k), \)
\[
  w = [(e, \Omega, \omega_K)],
\]
\( S^{n-32k} \) be the double point manifold of the immersion \( e, \omega_Q \times \omega_{\dot{Q}} : S^{n-32k} \to K(Q, 1) \times K(\dot{Q}, 1) \)
be the biquaternionic map.

Recall that \( n - 32k = \dim(S) \geq 14 \). Let \( i_T : T^{14} \subset S^{n-32k} \)
be a closed submanifold dual to the cohomology class
\[
  (\omega_S;Q \omega_S;\dot{Q})^{n-32k-14} \in H^{n-32k-14}(S^{n-32k}, Z/2),
\]
where \( \omega_S;Q = \omega_Q^*(\zeta_Q); \omega_S;\dot{Q} = \omega_{\dot{Q}}^*(\zeta_{\dot{Q}}) \in H^4(S^{n-32k}; Z/2). \)
The following (non-standard) representation $\chi_- : \mathbb{Q} \rightarrow \mathbb{Z}/2[3]$ transforms the quaternions $i, j, k$ into the following matrices:

$$i = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$
\[ j = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}, \]
\[ k = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \]
Let us define the vector bundles $\zeta_+, \zeta_-$ be over $S^{15}/\mathbb{Q}_a$.
The bundle $\zeta_+$ is defined by means of the representation $\chi_+$. The bundle $\zeta_-$ is defined by means of the representation $\chi_-$. The bundle $\zeta_+$ admits a complex structure. Note that $c_1(\zeta_+) = 0$, because the restriction of the bundle $\zeta_+$ over $S^3/\mathbb{Q} \subset S^{15}/\mathbb{Q}$ is the trivial complex bundle and $H^2(S^{15}/\mathbb{Q}; \mathbb{Z}) \to H^2(S^3/\mathbb{Q}; \mathbb{Z})$ is an isomorphism.
Therefore,

\[ p_1(2\zeta_+) = c_1^2(2\zeta_+) - 2c_2(2\zeta_+) = \]

\[ 4c_1^2(\zeta_+) - 4c_2(\zeta_+) = 4\zeta_Q \in H^4(K(Q,1);\mathbb{Z}). \]

By the analagous computation:

\[ p_1(2\dot{\zeta}_+) = 4\dot{\zeta}_Q \in H^4(K(\dot{Q},1);\mathbb{Z}). \]
The bundle $\zeta_-$ admits a complex structure. Note that $c_1(\zeta_-) = 0$ by analogical calculations. Therefore,

$$p_1(\zeta_+ \oplus \zeta_-) = c_1^2(\zeta_+ \oplus \zeta_-) - 2c_2(\zeta_+ \oplus \zeta_-) =$$

$$c_1^2(\zeta_+) + c_1^2(\zeta_-) + 2c_1(\zeta_+)c_1(\zeta_-) - 2c_2(\zeta_+) - 2c_2(\zeta_-) = 0,$$

because the Euler classes $e(\zeta_+) \in H^4(S^{15}/\mathbb{Q}; \mathbb{Z})$

$e(\zeta_-) \in H^4(S^{15}/\mathbb{Q}; \mathbb{Z})$ are opposite: $e(\zeta_+) = -e(\zeta_-)$. 
The normal bundle $\nu_T$ is stably isomorphic to the bundle $l\zeta_{T,+} \oplus l\dot{\zeta}_{T,+}$, where $l$ is an integer, $l \equiv 2 \pmod{4}$.

The bundle $\zeta_{T,+}$ is the 4-dimensional vector bundle over $T$ defined as

$$
\zeta_{T,+} = \omega_{T,Q}^*(\zeta_+), \\
\omega_{T,Q} = \omega_Q|_T : T^{14} \to K(Q,1).
$$

The bundle $\dot{\zeta}_{T,+}$ is the 4-dimensional vector bundle over $T$ defined as

$$
\dot{\zeta}_{T,+} = \omega_{T,Q}^*(\zeta_+), \\
\omega_{T,Q} = \omega_{\dot{Q}}|_T : T^{14} \to K(\dot{Q},1).
$$
Put $-T^{14}$ to be $T^{14}$ with the opposite orientation. The normal bundle $\nu_{-T}$ is stably isomorphic to the bundle $(l - 1)\zeta_{-T,+} \oplus \zeta_{-T,-} \oplus l\zeta_{-T,+}$ (we will put after $l = 2$ for the shortness).

The bundle $\zeta_{-T,+}$ is the 4-dimensional vector bundle defined as

$$\zeta_{-T,+} = \omega_{-T,Q}(\zeta_+),$$

$$\omega_{-T,Q} = \omega_Q|_{-T} : -T^{14} \to K(Q,1).$$

The bundle $\zeta_{-T,-}$ is the 4-dimensional vector bundle defined as

$$\zeta_{-T,-} = \omega_{-T,Q}(\zeta_-).$$
The bundle $\zeta_{-T,+}$ is the 4-dimensional vector bundle defined as

$$\dot{\zeta}_{-T,+} = \omega_{-T,\dot{Q}}(\zeta_+),$$

$$\omega_{-T,\dot{Q}} = \omega_{-T,\dot{Q}}|_{-T} : -T^{14} \to K(\dot{Q}, 1).$$
Let us assume that $\Theta_{\mathbb{Z}/5}(w) = 1$. Then the decomposition of the cycle $\omega_{\mathbb{Q} \oplus \hat{\mathbb{Q}}([T])}$ in the standard base of $H_{14}(\mathbb{Q} \oplus \hat{\mathbb{Q}}; \mathbb{Z})$ involves the element $u_7 \otimes v_7$, where $u_7 \in H_7(K(\mathbb{Q}, 1); \mathbb{Z}) = \mathbb{Z}/8$, $v_7 \in H_7(K(\hat{\mathbb{Q}}, 1); \mathbb{Z}) = \mathbb{Z}/8$ are the generators, $H_7(K(\mathbb{Q}, 1); \mathbb{Z}) \otimes H_7(K(\hat{\mathbb{Q}}, 1); \mathbb{Z}) \subset H_{14}(K(\mathbb{Q} \times \hat{\mathbb{Q}}, 1); \mathbb{Z})$. 
Let 

\[ F = id \cup -id : T^{14} \cup -T^{14} \to T^{14} \]

be the standard degree 0 map. Let us consider the following homology class:

\[ \mathcal{R} = (\omega_{Q \times \dot{Q}} \circ F)_*(\nu_T)_{\text{op}} + [p_1(\nu_{-T})]_{\text{op}}) \in \]

\[ H_{10}(K(Q \times \dot{Q}, 1); \mathbb{Z}), \]

where the upper index "op" stands for Poincaré dual.
Let us prove that $\aleph$ involves the element $4u_3 \otimes v_7 \in H_3(K(Q, 1); \mathbb{Z}) \otimes H_7(K(\hat{Q}, 1); \mathbb{Z}) \subset H_{10}(K(Q \times \hat{Q}, 1); \mathbb{Z})$.

Without loss of the generality we may assume that

$$\omega_{Q \times \hat{Q}, *}(\{T\}) = u_7 \otimes v_7 + xu_3 \otimes v_{11} + \ldots,$$

where $x$ is an arbitrary integer. (For all last terms in this formula the characteristic class $\aleph$ does not involve the element $u_3 \otimes v_7$ by the dimension reason). Under this assumption by the computation above we get:

$$F_*(\{p_1(\nu_T)\}^{op}) = 4u_3 \otimes v_7 + 4xu_3 \otimes v_7 + \ldots \in H_3(K(Q, 1); \mathbb{Z}) \otimes H_7(K(\hat{Q}, 1); \mathbb{Z}) \subset H_{10}(K(Q \times \hat{Q}, 1); \mathbb{Z}),$$

$$F_*(\{p_1(\nu_{-T})\}^{op}) = 4xu_3 \otimes v_7 + \ldots.$$
Therefore the first (normal) Pontrjagin class satisfy the equation:

\[ 0 \neq 4u_3 \otimes v_7 + \cdots = (\omega_{Q \times \bar{Q}} \circ F)_*([p_1(\nu_T)]^{op} + [p_1(\nu_{-T})]^{op}). \]

In particular, \( F \) is not cobordant to zero. But the mapping \( F \) is cobordant to zero by definition. Contradiction. Therefore \( \Theta_{\mathbb{Z}/5}(w) = 0. \)